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Tutorial 1: Functions, Derivatives & Engineering Applications

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Question 1

Evaluate the limit for the following if it exists:

(a) $\lim\limits_{x\to 5} \dfrac{x^2-25}{x^2+x-30}$

(b) $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

(c) $\lim\limits_{x\to 0} \dfrac{x}{3-\sqrt{x+9}}$

Solution

(a)

$$\begin{aligned}\lim_{x\to 5} \frac{x^2-25}{x^2+x-30} &= \lim_{x\to 5} \frac{(x-5)(x+5)}{(x-5)(x+6)} \\ &= \lim_{x\to 5} \frac{(x+5)}{(x+6)} \\ &= \frac{(5+5)}{(5+6)} \\ &= \frac{10}{11}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{(x-9)}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x\to 9} \frac{1}{(\sqrt{x}+3)} \\ &= \frac{1}{(\sqrt{9}+3)} \\ &= \frac{1}{6}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0} \frac{x}{3-\sqrt{x+9}} &= \lim_{x\to 0} \frac{x}{(3-\sqrt{x+9})} \frac{(3+\sqrt{x+9})}{(3+\sqrt{x+9})} \\ &= \lim_{x\to 0} \frac{x(3+\sqrt{x+9})}{9-(x+9)} \\ &= \lim_{x\to 0} \frac{x(3+\sqrt{x+9})}{-x} \\ &= \lim_{x\to 0} \frac{3+\sqrt{x+9}}{-1} \\ &= -6\end{aligned}$$

Question 2

Find the derivative of $\log(4+\cos x)$.

Solution

$$\begin{aligned} D(\log x) &= \frac{1}{x}\log e = \frac{1}{x\ln 10} \\ D[\log(4+\cos x)] &= \frac{1}{4+\cos x}(\log e)D(4+\cos x) \\ &= \frac{1}{4+\cos x}(\log e)(-\sin x) \\ &= \frac{-(\log e)(\sin x)}{4+\cos x} \end{aligned}$$

Question 3

Find $\frac{dy}{dx}$ for $\cos(x^2) = xe^y$.

Solution

$$\begin{aligned} \cos\left(x^2\right) &= xe^y \\ -\sin\left(x^2\right)\cdot 2x &= xe^y\frac{dy}{dx} + e^y \\ xe^y\frac{dy}{dx} &= -2x\sin\left(x^2\right) - e^y \\ \frac{dy}{dx} &= \frac{-2x\sin\left(x^2\right) - e^y}{xe^y} \end{aligned}$$

Question 4

Find $\frac{dy}{dx}$ for $x^3y^3 - 2y = x$.

Solution

$$\begin{aligned} x^3\left[3y^2\left(\frac{dy}{dx}\right)\right] + y^3\left[3x^2\right] - 2\left(\frac{dy}{dx}\right) &= 1 \\ 3x^3y^2\left(\frac{dy}{dx}\right) + 3x^2y^3 - 2\left(\frac{dy}{dx}\right) &= 1 \\ 3x^3y^2\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx}\right) &= 1 - 3x^2y^3 \\ \frac{dy}{dx} &= \frac{1-3x^2y^3}{3x^3y^2-2} \end{aligned}$$

Question 5

A curve in the plane is defined parametrically by the equations $x = 7\ln(t)$ and $y = \sqrt{1-4t}$. Find $\frac{dy}{dx}$.

Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\ &= \frac{\frac{d}{dt}(\sqrt{1-4t})}{\frac{d}{dt}(7\ln(t))} \\ &= \frac{\frac{1}{2}(1-4t)^{-\frac{1}{2}}\cdot(-4)}{\frac{7}{t}} \\ &= -\frac{2t}{7\sqrt{1-4t}} \end{aligned}$$

Question 6

A curve in the plane is defined parametrically by the equations $x = t^2 - 1$ and $y = 2e^t$. Find $\frac{dy}{dx}$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\ &= \frac{\frac{d}{dt}\left(2e^t\right)}{\frac{d}{dt}\left(t^2-1\right)} \\ &= \frac{2e^t}{2t} = \frac{e^t}{t}\end{aligned}$$

Question 7

Find y' for each of the following:

(a) $x^2\tan(y) + y^{10}\sec(x) = 2x$

(b) $x^3y^5 + 3x = 8y^3 + 1$

(c) $e^{2x+3y} = x^2 - \ln(xy^3)$

Solution

(a)

$$\begin{aligned}2x\tan(y) + x^2\sec^2(y)y' + 10y^9y'\sec(x) + y^{10}\sec(x)\tan(x) &= 2 \\ \left(x^2\sec^2(y) + 10y^9\sec(x)\right)y' &= 2 - y^{10}\sec(x)\tan(x) - 2x\tan(y) \\ y' &= \frac{2 - y^{10}\sec(x)\tan(x) - 2x\tan(y)}{x^2\sec^2(y) + 10y^9\sec(x)}\end{aligned}$$

(b)

$$\begin{aligned}3x^2y^5 + 5x^3y^4y' + 3 &= 24y^2y' \\ 3x^2y^5 + 3 &= 24y^2y' - 5x^3y^4y' \\ y' &= \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4}\end{aligned}$$

(c)

$$\begin{aligned}2e^{2x+3y} + 3y'e^{2x+3y} &= 2x - \frac{y^3}{xy^3} - \frac{3xy^2y'}{xy^3} \\ 2e^{2x+3y} + 3y'e^{2x+3y} &= 2x - \frac{1}{x} - \frac{3y'}{y} \\ \left(3e^{2x+3y} + 3y^{-1}\right)y' &= 2x - x^{-1} - 2e^{2x+3y} \\ y' &= \frac{2x - x^{-1} - 2e^{2x+3y}}{3e^{2x+3y} + 3y^{-1}}\end{aligned}$$

Question 8

Solve for y' if $y = \ln(\cos x^2)$.

Solution Let $u = \cos x^2$, then $\frac{du}{dx} = \left(-\sin x^2\right)(2x)$, $y = \ln u$, then $\frac{dy}{du} = \frac{1}{u}$

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{1}{u}\right)\left(-\sin x^2\right)(2x) \\ &= -\frac{2x\sin x^2}{\cos x^2} \\ &= -2x\tan x^2\end{aligned}$$

Question 9

Find y' for $10e^{2xy} = e^{15y} + e^{13x}$.

Solution

$$\begin{aligned}10e^{2xy} &= e^{15y} + e^{13x} \\ 10e^{2xy}\left(2xy' + 2y\right) &= e^{15y}\left(15y'\right) + e^{13x}(13) \\ 10e^{2xy}\left(2xy' + 2y\right) &= 15y'e^{15y} + 13e^{13x} \\ \left(20xe^{2xy} - 15e^{15y}\right)y' &= 13e^{13x} - 20ye^{2xy} \\ y' &= \frac{13e^{13x} - 20ye^{2xy}}{20xe^{2xy} - 15e^{15y}}\end{aligned}$$

Question 10

Solve $f'(x)$ if $f(x) = 2x(\arctan 5x)^2 + 6\tan(\cos 6x)$.

Solution $f(x) = 2x(\arctan 5x)^2 + 6\tan(\cos 6x)$ $\frac{d}{dx}\left(\tan^{-1} 5x\right)^2 = 2\tan^{-1} 5x\left(\frac{1}{1+(5x)^2}\right)(5) = \frac{10\tan^{-1} 5x}{1+25x^2}$

$$\begin{aligned}\frac{d}{dx}2x\left(\tan^{-1} 5x\right)^2 &= 2x\left(\frac{10\tan^{-1} 5x}{1+25x^2}\right) + \left(\tan^{-1} 5x\right)^2(2) \\ &= \frac{20x\tan^{-1} 5x}{1+25x^2} + 2\left(\tan^{-1} 5x\right)^2\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}6\tan(\cos 6x) &= 6\sec^2(\cos 6x)\frac{d}{dx}\cos 6x \\ &= 6\sec^2(\cos 6x) \times (-\sin 6x) \times 6 \\ &= -36\left(\sec^2(\cos 6x)\right)\sin 6x\end{aligned}$$

Therefore, $f'(x) = \frac{20x\tan^{-1} 5x}{1+25x^2} + 2\left(\tan^{-1} 5x\right)^2 - 36\left(\sec^2(\cos 6x)\right)\sin 6x$

Question 11

Solve y' if $y = 4x\sinh^{-1}(\frac{x}{6}) + \tanh^{-1}(\cos 10x)$.

Solution

$$\frac{d}{dx}4x\sinh^{-1}\left(\frac{x}{6}\right) = 4x\left[\frac{1}{\sqrt{1+\left(\frac{x}{6}\right)^2}}\cdot\frac{1}{6}\right] + 4\sinh^{-1}\left(\frac{x}{6}\right)$$

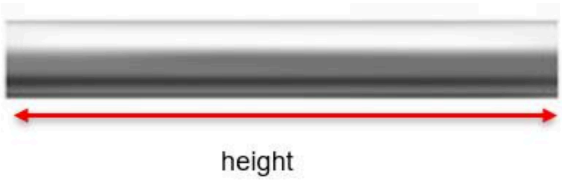
$$= \frac{\frac{2x}{3}}{\sqrt{1+\frac{x^2}{36}}} + 4\sinh^{-1}\left(\frac{x}{6}\right)$$

$$\begin{aligned}\frac{d}{dx}\tanh^{-1}(\cos 10x) &= \frac{1}{1-(\cos 10x)^2}\left(\frac{d}{dx}\cos 10x\right)\\ &= \frac{1}{1-(\cos 10x)^2}(-10\sin 10x)\\ &= \frac{-10\sin 10x}{1-\cos^2 10x}\\ &= -\frac{10\sin 10x}{\sin^2 10x}\\ &= -\frac{10}{\sin 10x}\\ &= -10\csc 10x\end{aligned}$$

$$\therefore f'(x) = \frac{\frac{2x}{3}}{\sqrt{1+\frac{x^2}{36}}} + 4\sinh^{-1}(\frac{x}{6}) - 10\csc 10x$$

Question 12

The engineering discipline of piping design studies the efficient transport of fluid. You are required to design a stable pipe that can hold 200 litres of fluid at a time. Assuming that it is a closed ended pipe at both ends, determine the dimensions (radius and height in cm) of the pipe that will minimize the amount of material used to construct the pipe with justification.





Solution

$$A = 2\pi r^2 + 2\pi rh$$

$$\pi r^2 h = 200,000 \quad \Rightarrow h = \frac{200,000}{\pi r^2}$$

$$A(r) = 2\pi r^2 + \frac{400,000}{r}$$

$$A'(r) = 4\pi r - \frac{400,000}{r^2}$$

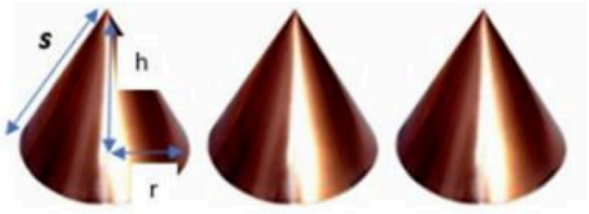
Find critical number, $A'(r) = 0$, Radius, $r = 31.7$ cm.

$A''(r) = 4\pi + \frac{800,000}{r^3} \rightarrow$ positive $\rightarrow$ concavity: concave upward $\rightarrow$ min point.

Hence $h = 2r \approx 63.4$ cm.

Question 13

You are a manufacturer of metal cones. In order to minimize the material used in the manufacturing process, you need to fabricate the metal cones shown below. You are provided with 3 copper cones with total volume of $15m^3$. Each cone consists of the flat base surface and lateral surface. Determine the possible dimensions of the lateral surface for each cone to estimate the amount of copper material needed to construct each cone's lateral surface. (Given the dimensions of cone are height of the cone, h, the slant length, s, and the radius of the base, r).



Solution Volume of each cone $= \frac{1}{3}\pi r^2 h = 5, \quad h = \frac{15}{\pi r^2}$
Area, A(r) in terms of r by substituting h.

$$\mathrm{A}(r) = \pi r s \quad \text{where } s \text{ is the slant length,}$$
$$\mathrm{A}(r) = \pi r \left(\sqrt{r^2 + h^2}\right)$$
$$\mathrm{A}(r) = \pi r \left(\sqrt{r^2 + \left(\frac{15}{\pi r^2}\right)^2}\right)$$

Since $\pi r s = \pi r\sqrt{r^2 + \dfrac{225}{\pi^2 r^4}} = \dfrac{\sqrt{\pi^2 r^6 + 225}}{r}$, differentiate exactly:

$$A'(r) = \frac{2\pi^2 r^6 - 225}{r^2\sqrt{\pi^2 r^6 + 225}}.$$

Because $r > 0$ and the denominator is positive, $A'(r) = 0$ gives

$$2\pi^2 r^6 = 225, \qquad r = \left(\frac{225}{2\pi^2}\right)^{1/6} \approx 1.5002 \text{ m}.$$

The area tends to infinity as $r \to 0^+$ or $r \to \infty$, so this critical point is the global minimum.
Hence, dimension :

- Radius, $r \approx 1.5002$ m,
- Height, $h = \dfrac{15}{\pi r^2} \approx 2.1216$ m
- Slant length, $s = \sqrt{r^2 + h^2} \approx 2.5984$ m

  Note. The calculation above minimises only the **lateral** surface, because that is the quantity requested in the final sentence of the question. If the flat base is also part of the material cost, use $A = \pi r^2 + \pi r s$; that interpretation gives approximately $r = 1.1907$ m, $h = 3.3678$ m, and $s = 3.5721$ m.

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