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Tutorial 10: Multiple Integrals

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Question 1

Evaluate the following:

a. $\displaystyle\int_1^2 \int_2^3 (2x - y)dydx$

b. $\displaystyle\int_1^5 \int_{-1}^2 (x - 5y)dxdy$

c. $\displaystyle\int_1^3 \int_2^5 (2x - 3y)dxdy$

Solution

a.

$$\begin{aligned}
\int_1^2 \int_2^3 (2x - y)dydx &= \int_1^2 \left[(2x)y - \left(\frac{y^2}{2}\right)\right]_2^3 dx \\
&= \int_1^2 \left[\left((2x)(3) - \frac{3^2}{2}\right) - \left((2x)(2) - \frac{2^2}{2}\right)\right] dx \\
&= \int_1^2 [(6x - 4.5) - (4x - 2)]dx = \int_1^2 2x - 2.5dx \\
&= \left[x^2 - 2.5x\right]_1^2 = [(4 - 5) - (1 - 2.5)] \\
&= 4 - 5 - 1 + 2.5 = 0.5
\end{aligned}$$

b.

$$\begin{aligned}
\int_1^5 \int_{-1}^2 (x - 5y)dxdy &= \int_1^5 \left[\frac{x^2}{2} - (5y)x\right]_{-1}^2 dy \\
&= \int_1^5 [(2 - 10y) - (0.5 + 5y)]dy = \int_1^5 1.5 - 15ydy \\
&= \left[1.5y - \frac{15y^2}{2}\right]_1^5 = [(7.5 - 187.5) - (1.5 - 7.5)] = -174
\end{aligned}$$

c.

$$\begin{aligned}
\int_1^3 \int_2^5 (2x - 3y)dxdy &= \int_1^3 \left[\frac{2x^2}{2} - (3y)x\right]_2^5 dy \\
&= \int_1^3 \left[\left(5^2 - 3(5)y\right) - \left(2^2 - 3(2)y\right)\right] dy \\
&= \int_1^3 [21 - 9y]dy \\
&= \left[21y - \frac{9y^2}{2}\right]_1^3 \\
&= \left[\left(21(3) - \frac{9(3)^2}{2}\right) - \left(21(1) - \frac{9(1)^2}{2}\right)\right] \\
&= (63 - 40.5) - (21 - 4.5) = 6
\end{aligned}$$

Question 2

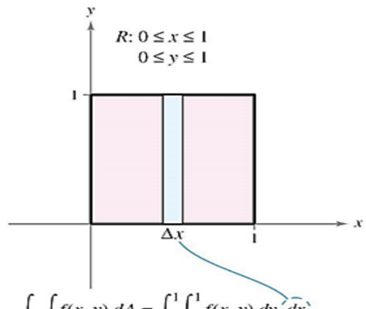
Evaluate each of the following integrals over the given region D :

a. $\iint_D \left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right) dA$ where D is the region given by $0 \le x \le 1, 0 \le y \le 1$

b. $\iint_D \left(4xy-y^3\right) dA$ where D is the region given by $y = x^{0.5}$ and $y = x^3$

Solution

a.



$\int_R \int f(x,y)\, dA = \int_0^1 \int_0^1 f(x,y)\, dy\, dx$

The volume of the solid region is $\frac{2}{3}$.

- Because the region R is a square, it is both vertically and horizontally simple, and you can use either order of integration.
- Choose $dy\ dx$ by placing a vertical representative rectangle in the region, as shown in the figure at the right.

$$
\begin{aligned}
\iint_R \left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right) dA &= \int_0^1 \int_0^1 \left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right) dydx \\
&= \int_0^1 \left[\left(1-\frac{1}{2}x^2\right)y-\frac{y^3}{6}\right]_0^1 dx \\
&= \int_0^1 \left(\frac{5}{6}-\frac{1}{2}x^2\right) dx \\
&= \left[\frac{5}{6}x-\frac{x^3}{6}\right]_0^1 = \frac{2}{3}
\end{aligned}
$$

b.

$$
\iint_D \left(4xy-y^3\right) dA \quad 0 \le x \le 1 \text{ and } x^3 \le y \le x^{0.5}
$$

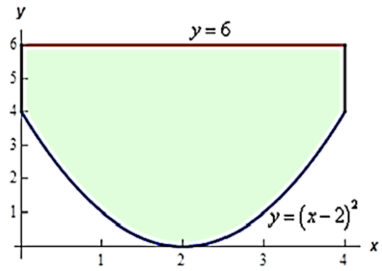
$$
\begin{aligned}
&= \int_0^1 \int_{x^3}^{x^{0.5}} 4xy-y^3 dydx \\
&= \int_0^1 \left[\frac{4xy^2}{2}-\frac{y^4}{4}\right]_{x^3}^{x^{0.5}} dx \\
&= \int_0^1 \left[\left(\frac{4x\left(x^{0.5}\right)^2}{2}-\frac{\left(x^{0.5}\right)^4}{4}\right)-\left(\frac{4x\left(x^3\right)^2}{2}-\frac{\left(x^3\right)^4}{4}\right)\right] dx \\
&= \int_0^1 \left[\left(\frac{4x^2}{2}-\frac{x^2}{4}\right)-\left(\frac{4x^7}{2}-\frac{x^{12}}{4}\right)\right] dx \\
&= \int_0^1 \left[\frac{7}{4}x^2-2x^7+\frac{1}{4}x^{12}\right] dx \\
&= \left[\frac{7}{12}x^3-\frac{1}{4}x^8+\frac{1}{52}x^{13}\right]_0^1 \\
&= \frac{55}{156}
\end{aligned}
$$

Question 3

Evaluate the following integral:

$$\iint_D 42y^2 - 12x dA \quad \text{where } D = \left\{(x,y) \mid 0 \le x \le 4, (x-2)^2 \le y \le 6\right\}$$

Solution



$$\begin{aligned}
\iint_D 42y^2 - 12x dA &= \int_0^4 \int_{(x-2)^2}^6 42y^2 - 12x dy dx \\
&= \int_0^4 \left[14y^3 - 12xy\right]_{(x-2)^2}^6 dx \\
&= \int_0^4 3024 - 72x - 14(x-2)^6 + 12x(x-2)^2 dx \\
&= \int_0^4 3024 - 24x - 48x^2 + 12x^3 - 14(x-2)^6 dx \\
&= \left[3024x - 12x^2 - 16x^3 + 3x^4 - 2(x-2)^7\right]_0^4 \\
&= 11136
\end{aligned}$$

Question 4

Evaluate the following integral over the indicated rectangle (a) by integrating with respect to x first and (b) with respect to y first.

$$\iint_R 12x - 18y dA \quad R = [-1,4] \times [2,3]$$

Solution

(a)

$$\begin{aligned}
\iint_R 12x - 18y dA &= \int_2^3 \int_{-1}^4 12x - 18y dx dy \\
&= \int_2^3 \left[6x^2 - 18xy\right]_{-1}^4 dy = \int_2^3 90 - 90y dy \\
&= \left[90y - 45y^2\right]_2^3 = -135
\end{aligned}$$

(b)

$$\begin{aligned}
\iint_R 12x - 18y dA &= \int_{-1}^4 \int_2^3 12x - 18y dy dx \\
&= \int_{-1}^4 \left[12xy - 9y^2\right]_2^3 dx = \int_{-1}^4 12x - 45 dx \\
&= \left[6x^2 - 45x\right]_{-1}^4 = -135
\end{aligned}$$

Question 5

Compute the following double integral over the indicated rectangle

a. $\iint_R 6y\sqrt{x} - 2y^3 dA \quad R = [1,4] \times [0,3]$

b. $\iint_R ye^{y^2-4x} dA \quad R = [0,2] \times [0,\sqrt{8}]$

Solution

a.

$$\begin{aligned}
\iint_R 6y\sqrt{x} - 2y^3 dA &= \int_0^3 \int_1^4 6yx^{\frac{1}{2}} - 2y^3 dxdy \\
&= \int_0^3 \left(4yx^{\frac{3}{2}} - 2xy^3\right)\Big|_1^4 dy = \int_0^3 28y - 6y^3 dy \\
&= \left(14y^2 - \frac{3}{2}y^4\right)\Big|_0^3 = \frac{9}{2} \\
&= \int_0^2 \left(\frac{1}{2}e^{y^2-4x}\right)\Big|_0^{\sqrt{8}} dx = \int_0^2 \frac{1}{2}\left(e^{8-4x} - e^{-4x}\right) dx \\
&= \left(\frac{1}{2}\left(-\frac{1}{4}e^{8-4x} + \frac{1}{4}e^{-4x}\right)\right)\Big|_0^2 \\
&= \frac{1}{8}\left(e^8 + e^{-8} - 2\right) = 372.3698
\end{aligned}$$

b.

$$\int_R ye^{y^2-4x} dA \quad = \int_0^2 \int_0^{\sqrt{8}} ye^{y^2-4x} dydx \qquad u = y^2 - 4x \to du = 2ydyy$$

Question 6

Evaluate:

a. $\int_{-3}^{1} \int_{-1}^{2} \int_{0}^{1} 6xyzdydxdz$

b. $\int_{0}^{4} \int_{-2}^{-1} \int_{1}^{2} xydxdydz$

Solution

a.

$$\begin{aligned}
\int_{-3}^{1} \int_{-1}^{2} \int_{0}^{1} 6xyzdydxdz &= \int_{-3}^{1} \int_{-1}^{2} \left[3xy^2z\right]_0^1 dxdz \\
&= \int_{-3}^{1} \int_{-1}^{2} 3xzdxdz \\
&= \int_{-3}^{1} \left[\frac{3}{2}x^2z\right]_{-1}^{2} dz = \int_{-3}^{1} \left[6z - \frac{3}{2}z\right] dz \\
&= \int_{-3}^{1} \left[\frac{9}{2}z\right] dz = \left[\frac{9}{4}z^2\right]_{-3}^{1} \\
&= \frac{9}{4} - \frac{9}{4}(9) = \frac{9}{4} - \frac{81}{4} \\
&= \frac{-72}{4} = -18
\end{aligned}$$

b.

$$\int_0^4\int_{-2}^{-1}\int_1^2 xydxdydz = \int_0^4\int_{-2}^{-1}\left[\frac{x^2}{2}y\right]_1^2 dydz = \int_0^4\int_{-2}^{-1} 1.5ydydz$$
$$= \int_0^4\left[\frac{1.5y^2}{2}\right]_{-2}^{-1} dz = \int_0^4 -2.25dz$$
$$= [-2.25z]_0^4 = -9$$

Question 7

Evaluate:

a. $\iiint_B xyz^2dV \qquad \begin{array}{l} 0 \le x \le 1 \\ -1 \le y \le 2 \\ 0 \le z \le 3 \end{array}$

b. $\iiint_B 4x^2y - z^3dzdydx \qquad \begin{array}{l} 2 \le x \le 3 \\ -1 \le y \le 4 \\ 0 \le z \le 1 \end{array}$

Solution

a.

$$\iiint_B xyz^2dV = \int_0^3\int_{-1}^2\int_0^1 xyz^2dxdydz$$
$$= \int_0^3\int_{-1}^2\left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dydz = \int_0^3\int_{-1}^2 \frac{yz^2}{2}dydz$$
$$= \int_0^3\left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz = \int_0^3 \frac{3z^2}{4}dz$$
$$= \left[\frac{z^3}{4}\right]_0^3 = \frac{27}{4}$$

b.

$$\iiint_B 4x^2y - z^3dzdydx = \int_2^3\int_{-1}^4\left[4x^2yz - \frac{1}{4}z^4\right]_0^1 dydx$$
$$= \int_2^3\int_{-1}^4\left[4x^2yz - \frac{1}{4}z^4\right]_0^1 dydx$$
$$= \int_2^3\int_{-1}^4 \frac{1}{4} - 4x^2ydydx$$
$$= \int_2^3\left[\frac{1}{4}y - 2x^2y^2\right]_{-1}^4 dx = \int_2^3 \frac{5}{4} - 30x^2dx$$
$$= \left[\frac{5}{4}x - 10x^3\right]_2^3 = -\frac{755}{4}$$

Question 8

Integrate the function $(x, y, z) = xy$ over the volume enclose by the planes $z = x + y$ and $z = 0$, and between the surfaces $y = x^2$ and $x = y^2$.

Solution

Since z is expressed as a function of (x, y), the integration is done in the z direction first. Considering the xy-plane, the two curves meet at $(0, 0)$ and $(1, 1)$. The integral is:

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} f(x,y,z) dzdydx &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xydzdydx \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} xy[z]_{z=0}^{z=x+y} dydx \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x+y)dydx \int_0^1 \int_{x^2}^{\sqrt{x}} x^2y + y^2xdydx \\
&= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{y^3x}{3}\right]_{y=x^2}^{y=\sqrt{x}} dx \\
&= \frac{1}{2}\int_0^1 \left(x^2\sqrt{x}^2 - x^2\left(x^2\right)^2\right) dx \\
&= \frac{1}{2}\int_0^1 \left(x^3 - x^6\right) dx + \frac{1}{3}\int_0^1 \left(\sqrt{x}x - \left(x^2\right)^{\frac{5}{2}} - x^7\right) dx \\
&= \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 + \frac{1}{3}\left[\frac{2x^{\frac{7}{2}}}{7} - \frac{x^8}{8}\right]_0^1 \\
&= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) = \frac{3}{28}
\end{aligned}$$

Question 9

Find the mass for the following square lamina, represented by the unit square (shown below) with variable density $\rho(x, y) = (x + y + 2)\text{g/cm}^2$.

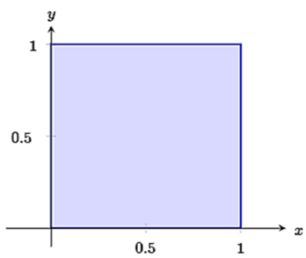


Figure 1. A region R representing a lamina

Solution

$$\begin{aligned}
M = \iint_R (x+y+2)dA &= \int_0^1 \int_0^1 (x+y+2)dxdy \\
&= \int_0^1 \left[\frac{1}{2}x^2 + x(y+2)\right]_0^1 dy \\
&= \int_0^1 \frac{5}{2} + ydy \\
&= \left[\frac{5}{2}y + \frac{y^2}{2}\right]_0^1 \\
&= \frac{5}{2} + \frac{1}{2} \\
&= 3g
\end{aligned}$$

Question 10

Find the mass and center of mass for the following solid region Q bounded by $x + 2y + 3z = 6$ and the coordinate planes (as shown below) and has density $\rho(x, y, z) = x^2yz$.

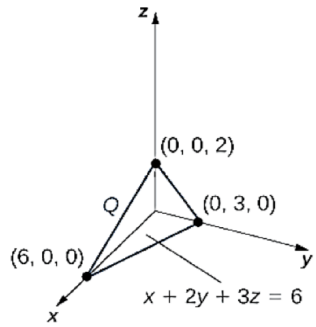


Figure 2: Finding the mass of a three-dimensional solid Q.

Solution

The region Q is a tetrahedron meeting the axes at the points $(6, 0, 0), (0, 3, 0)$ and $(0, 0, 2)$. To find the limits of integration, let $z = 0$ in the slated plane $z = \frac{1}{3}(6 - x - 2y)$.

Then for x and y, find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x + 2y = 6$. Therefore, the mass (m) is:

$$\begin{aligned} m = \iiint_Q \rho(x, y, z) dV &= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x_-} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2yzdzdydx \\ &= \frac{108}{35} \end{aligned}$$

To find the center of mass of Q, we need to find the moments about the xy-plane, the xz-plane and the yz-plane:

$$\begin{aligned} M_{xy} = \iiint_Q z\rho(x, y, z) dV &= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2yz^2dzdydx \\ &= \frac{54}{35} \end{aligned}$$

$$\begin{aligned} M_{xz} = \iiint_Q y\rho(x, y, z) dV &= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2y^2zdzdydx \\ &= \frac{81}{35} \end{aligned}$$

$$\begin{aligned} M_{yz} = \iiint_Q x\rho(x, y, z) dV &= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^3yzdzdydx \\ &= \frac{243}{35} \end{aligned}$$

Hence the center of mass is:

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{243/35}{108/35} = \frac{243}{108} \quad \bar{y} = \frac{M_{xz}}{m} = \frac{81/35}{108/35} = \frac{81}{108} \quad \bar{y} = \frac{M_{xy}}{m} = \frac{54/35}{108/35} = \frac{54}{108}$$

$$= 2.25 \qquad = 0.75 \qquad = 0.5$$

The center of mass: $\quad (2.25, 0.75, 0.5)$