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Tutorial 11: Multiple Integrals in Polar Coordinate & Its Engineering Application

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Question 1

In the following exercises, change the cartesian integral into an equivalent polar coordinate integral. Then solve the integral in polar coordinate:

a) $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$

b) $\displaystyle\int_{0}^{2}\int_{0}^{x} ydydx$

c) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}dydx$

d) $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e\sqrt{x^2+y^2}dxdy$

Solution

a)

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx = \int_{-1}^{1}\sqrt{1-x^2}dx$$

Let $x = \sin(\theta)$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, $dx = \cos(\theta)d\theta$.

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx &= \int_{-1}^{1}\sqrt{1-x^2}dx \\
&= \int_{-1}^{1}\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta \\
&= \int_{-1}^{1}\sqrt{\cos^2(\theta)}\cos(\theta)d\theta \\
&= \int_{-1}^{1}|\cos(\theta)|\cos(\theta)d\theta = \int_{-1}^{1}\cos^2(\theta) \\
&= \int_{-1}^{1}\frac{1}{2}+\frac{1}{2}\cos(2\theta)d\theta \\
&= \frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\Big|_{-1}^{1} \\
&= \frac{1}{2}\sin^{-1}(x)+\frac{1}{2}x\sqrt{1-x^2}\Big|_{-1}^{1} \\
&= \frac{\pi}{2}
\end{aligned}$$

b)

$$\begin{aligned}
\int_{0}^{2}\int_{0}^{x} ydydx &= \int_{0}^{2}\frac{x^2}{2}dx \\
&= \frac{x^3}{6}\Big|_{0}^{2} = \frac{4}{3}
\end{aligned}$$

c)

$$
\begin{aligned}
I &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} dydx \quad \text{(region: unit disk } x^2+y^2 \le 1\text{)} \\
&= \int_{0}^{2\pi}\int_{0}^{1} \frac{2}{(1+r^2)^2}, r, dr, d\theta \qquad \text{(polar coordinates } x = r\cos\theta, ; y = r\sin\theta\text{)} \\
&= \int_{0}^{2\pi}\left(\int_{0}^{1} \frac{2r}{(1+r^2)^2}, dr\right) d\theta
\end{aligned}
$$

Use the substitution $u = 1 + r^2$, so $du = 2r, dr$. When $r = 0 \Rightarrow u = 1$, and when $r = 1 \Rightarrow u = 2$. Thus,

$$
\begin{aligned}
\int_{0}^{1} \frac{2r}{(1+r^2)^2} dr &= \int_{1}^{2} \frac{1}{u^2} du \\
&= \left[-\frac{1}{u}\right]_1^2 \\
&= -\frac{1}{2} + 1 = \frac{1}{2}.
\end{aligned}
$$

Therefore,

$$
I = \int_{0}^{2\pi} \frac{1}{2} d\theta = \frac{1}{2} \cdot 2\pi = \pi
$$

d)

$$
\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy
$$

The limits of integration are given by $0 \le y \le \ln 2$ and $0 \le x \le \sqrt{(\ln 2)^2 - y^2}$. The upper limit for x can be rewritten as $x^2 \le (\ln 2)^2 - y^2$, which simplifies to $x^2 + y^2 \le (\ln 2)^2$. This inequality describes the area inside and on a circle centered at the origin with a radius of $R = \ln 2$.

Given that $x \ge 0$ and $y \ge 0$, the region of integration is the portion of this circle in the first quadrant.

The circular nature of the domain and the form of the integrand $(x^2 + y^2)$ make it ideal to convert to polar coordinates. The conversion formulas are:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
x^2 + y^2 &= r^2 \\
dx\, dy &= r\, dr\, d\theta
\end{aligned}
$$

The limits for the region in polar coordinates become:

- Radius: $0 \le r \le \ln 2$
- Angle: $0 \le \theta \le \frac{\pi}{2}$

The integrand $e^{\sqrt{x^2+y^2}}$ becomes $e^{\sqrt{r^2}} = e^r$.

Thus, the integral in polar coordinates is:

$$
\int_{0}^{\frac{\pi}{2}}\int_{0}^{\ln 2} e^r \cdot r\, dr\, d\theta
$$

Next, we evaluate the integral with respect to r using integration by parts, where $\int u\, dv = uv - \int v\, du$. Let $u = r$ and $dv = e^r\, dr$. Then $du = dr$ and $v = e^r$.

$$
\begin{aligned}
\int_0^{\ln 2} re^r \, dr &= \left[re^r - \int e^r \, dr\right]_0^{\ln 2} \\
&= [re^r - e^r]_0^{\ln 2} \\
&= [(r-1)e^r]_0^{\ln 2} \\
&= ((\ln 2 - 1)e^{\ln 2}) - ((0-1)e^0) \\
&= ((\ln 2 - 1) \cdot 2) - (-1 \cdot 1) \\
&= 2\ln 2 - 2 + 1 \\
&= 2\ln 2 - 1 = \ln(4) - 1
\end{aligned}
$$

Finally, we substitute the result from the inner integral into the outer integral and evaluate with respect to θ:

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} (\ln 4 - 1) \, d\theta &= (\ln 4 - 1) \int_0^{\frac{\pi}{2}} d\theta \\
&= (\ln 4 - 1) \, [\theta]_0^{\frac{\pi}{2}} \\
&= (\ln 4 - 1) \left(\frac{\pi}{2} - 0\right) \\
&= \frac{\pi}{2}(\ln 4 - 1)
\end{aligned}
$$

Question 2

Evaluate the $\iint 1 - x^2 - y^2 dA$ using polar coordinates

Solution

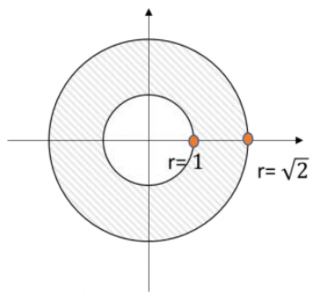
The region R is a unit circle, so we can describe it as $R = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.
Using the conversion $x = r\cos\theta, y = r\sin\theta$, and $dA = rdrd\theta$, we have

$$
\begin{aligned}
\iint_R \left(1 - x^2 - y^2\right) dA &= \int_0^{2\pi} \int_0^1 \left(1 - r^2\right) rdrd\theta \\
&= \int_0^{2\pi} \int_0^1 \left(r - r^3\right) drd\theta \\
&= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta \\
&= \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2}
\end{aligned}
$$

Question 3

Find the volume below $z = \dfrac{y^2}{x^2 + y^2}$, above xy-plane and between cylinder $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$

Solution



$$
\begin{aligned}
\int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} z\, r\, dr\, d\theta &= \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} \frac{y^2}{x^2+y^2} r\, dr\, d\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} \frac{y^2}{r^2} r\, dr\, d\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} \frac{r^2 \sin^2\theta}{r^2} r\, dr\, d\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}} \sin^2\theta\, r\, dr\, d\theta \\
&= \int_{\theta=0}^{2\pi} \left(\frac{r^2}{2}\sin^2\theta\right)\Bigg|_{r=1}^{r=\sqrt{2}} d\theta \\
&= \int_{\theta=0}^{2\pi} \frac{\sin^2\theta}{2}\, d\theta \\
&= \frac{1}{2}\int_{\theta=0}^{2\pi} \frac{1}{2}\left(1-\cos 2\theta\right)\, d\theta \\
&= \frac{1}{4}\int_{\theta=0}^{2\pi} (1-\cos 2\theta)\, d\theta \\
&= \frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_{\theta=0}^{\theta=2\pi} \\
&= \frac{1}{4}(2\pi) = \frac{\pi}{2}
\end{aligned}
$$

Question 4

Find the volume between the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$

Solution
We seek the volume inside the sphere $x^2+y^2+z^2=1$ and above the cone $z=\sqrt{x^2+y^2}$.
In spherical coordinates (ρ,ϕ,θ), $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$, the volume element is

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

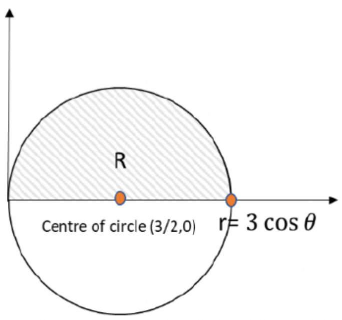
.
The cone $z=\sqrt{x^2+y^2}$ becomes $\rho\cos\phi=\rho\sin\phi \quad\Rightarrow\quad \cos\phi=\sin\phi \quad\Rightarrow\quad \phi=\frac{\pi}{4}$.
Thus, the region is $0\le\theta\le 2\pi,\quad 0\le\phi\le\frac{\pi}{4},\quad 0\le\rho\le 1$.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\ d\rho\, d\phi\, d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/4}\sin\phi\, d\phi\right)\left(\int_0^1 \rho^2\, d\rho\right) \\
&= (2\pi)\left[-\cos\phi\right]_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^1 \\
&= 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{3} \\
&= \frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right) = \frac{2\pi}{3} - \frac{\pi\sqrt{2}}{3} = \frac{\pi}{3}\left(2-\sqrt{2}\right)
\end{aligned}
$$

Question 5

Volume is equal to area only if the height (z) is equal to 1 . Find the area of R where R is the region bound by $r = 3\cos\theta$.

Solution



Region bound by $r = 3\cos\theta$

How to sketch the region R?

$$
\begin{aligned}
r \cdot r &= 3\cos\theta \cdot r \\
r^2 &= 3r\cos\theta \\
x^2 + y^2 &= 3r\cos\theta \\
x^2 + y^2 &= 3x \\
x^2 - 3x + y^2 &= 0 \\
\left(x - \tfrac{3}{2}\right)^2 + y^2 &= \left(\tfrac{3}{2}\right)^2 = \tfrac{9}{4}
\end{aligned}
$$

$$x = \tfrac{3}{2}, \quad y = 0$$

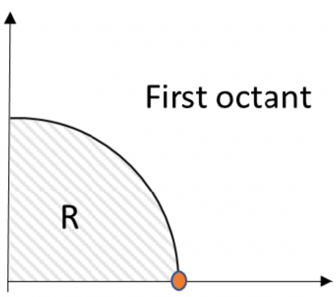
Volume = area when $z = 1$. Hence,

$$
\begin{aligned}
A &= \iint dA \\
&= 2\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{r=3\cos\theta} r\, dr\, d\theta \\
&= \int_{\theta=0}^{\pi} \int_{r=0}^{r=3\cos\theta} r\, dr\, d\theta \quad \Leftarrow \text{if we set up like this, we will get } \cos\theta = -1 \text{ and } \cos 0 = 1. \text{ The total will be 0.} \\
&= 2\int_{\theta=0}^{\pi/2} \left.\frac{r^2}{2}\right|_{r=0}^{r=3\cos\theta} d\theta \\
&= \int_{\theta=0}^{\pi/2} 9\cos^2\theta\, d\theta \\
&= \int_{\theta=0}^{\pi/2} \frac{9}{2}(1 + \cos 2\theta)\, d\theta = \frac{9\pi}{4}
\end{aligned}
$$

Question 6

$\iiint y dV$, a solid is bound by $z = 4 - x^2 - y^2$ in the first octant $(x = 0, y = 0, z = 0)$

Solution

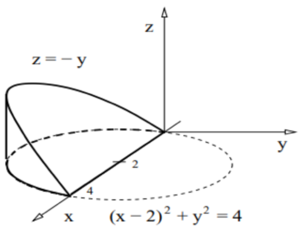


$$
\begin{aligned}
z &= 4 - x^2 - y^2 \\
0 &= 4 - x^2 - y^2 \\
2^2 &= x^2 + y^2, \quad r = 2
\end{aligned}
$$

$$
\begin{aligned}
\iiint ydV &= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{4-x^2-y^2} y\, dz\, r dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 [yz]_{z=0}^{z=4-x^2-y^2}\, r\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 (4-x^2-y^2) y\, r\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 (4-r^2) r \sin\theta\, r\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 \sin\theta \cdot r^2(4-r^2)\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 (4r^2 \sin\theta - r^4 \sin\theta)\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{4r^3}{3}\sin\theta - \frac{r^5}{5}\sin\theta\right]_{r=0}^{r=2} d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{32}{3}\sin\theta - \frac{32}{5}\sin\theta\right) d\theta \\
&= \left[-\frac{32}{3}\cos\theta + \frac{32}{5}\cos\theta\right]_0^{\frac{\pi}{2}} \\
&= \left(-\frac{32}{3}\cos\frac{\pi}{2} + \frac{32}{5}\cos\frac{\pi}{2}\right) - \left(-\frac{32}{3}\cos 0 + \frac{32}{5}\cos 0\right) \\
&= (0) - \left(-\frac{32}{3} + \frac{32}{5}\right) \\
&= \frac{32}{3} - \frac{32}{5} \\
&= \frac{160-96}{15} \\
&= \frac{64}{15}
\end{aligned}
$$

Question 7

Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $\left(x^2-2\right)^2 + y^2 = 4$ by the planes $z = 0z = 0$ and $z = -y$.



Solution
First, sketch the integration region.

$(x-2)^2 + y^2 = 4$ is a circle, since $x^2 + y^2 = 4x \Leftrightarrow r^2 = 4r\cos(\theta)$

$$r = 4\cos(\theta)$$

Since $0 \leq z \leq -y$, the integration region is on the $y \leq 0$ part of the $z = 0$ plane.

$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \int_0^{-r\sin(\theta)} r\, dz\, dr\, d\theta.$$

$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} [-r\sin(\theta) - 0]\, r\, dr\, d\theta$$

$$V = -\int_{3\pi/2}^{2\pi} \left(\left[\frac{r^3}{3} \right]_0^{4\cos(\theta)} \right) \sin(\theta)\, d\theta.$$

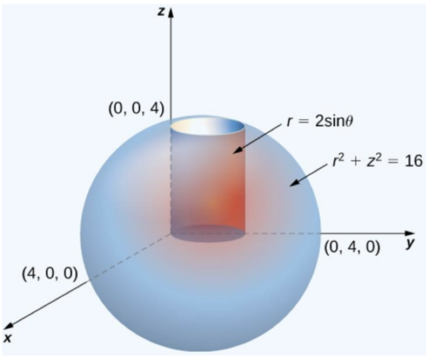
$$V = -\int_{3\pi/2}^{2\pi} \frac{4^3}{3} \cos^3(\theta) \sin(\theta)\, d\theta.$$

Introduce the substitution: $u = \cos(\theta)$, $du = -\sin(\theta)\, d\theta$;

$$V = \frac{4^3}{3} \int_0^1 u^3\, du = \frac{4^3}{3} \left(\frac{u^4}{4} \bigg|_0^1 \right) = \frac{4^3}{3} \cdot \frac{1}{4} = \frac{16}{3}$$

Question 8

Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin. Set up a triple integral over this region with a function $f(r, \theta, z)$ in cylindrical coordinates.



Solution

Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin. Set up a triple integral over this region for a function $f(r, \theta, z)$ in cylindrical coordinates.

The triple integral in cylindrical coordinates for a function $f(r, \theta, z)$ is given by:

$$\iiint_E f(r, \theta, z)\, dV = \int_{\theta_{min}}^{\theta_{max}} \int_{r_{min}(\theta)}^{r_{max}(\theta)} \int_{z_{min}(r,\theta)}^{z_{max}(r,\theta)} f(r, \theta, z)\, r\, dz\, dr\, d\theta$$

We need to find the bounds for z, r, and θ that describe the region E.

The region is bounded below by the $r\theta$-plane, which corresponds to the plane $z = 0$. The region is bounded above by the sphere of radius 4 centered at the origin. The equation of this sphere in Cartesian coordinates is $x^2 + y^2 + z^2 = 16$. In cylindrical coordinates, since $r^2 = x^2 + y^2$, this becomes $r^2 + z^2 = 16$.

Solving for z, we get $z = \sqrt{16 - r^2}$ (we take the positive root for the upper hemisphere). Therefore, the limits for z are:

$$0 \leq z \leq \sqrt{16 - r^2}$$

The projection of the volume onto the $r\theta$-plane (the xy-plane) is determined by the cylinder $r = 2\sin\theta$. For any given angle θ, the radius r extends from the origin ($r = 0$) to the edge of the cylinder ($r = 2\sin\theta$). Therefore, the limits for r are:

$$0 \le r \le 2\sin\theta$$

To find the limits for θ, we analyze the base of the cylinder $r = 2\sin\theta$ in the xy-plane. To ensure the radius r is non-negative, we must have $\sin\theta \ge 0$. This condition is met for angles in the first and second quadrants.

The curve starts at the origin when $\theta = 0$ (since $r = 2\sin(0) = 0$), traces a full circle, and returns to the origin when $\theta = \pi$ (since $r = 2\sin(\pi) = 0$). Therefore, the limits for θ are:

$$0 \le \theta \le \pi$$

Combining these limits, the triple integral is set up as follows:

$$\int_0^{\pi} \int_0^{2\sin\theta} \int_0^{\sqrt{16-r^2}} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

Question 9

Find the volume of solid bound by $z = 2$ and $z = \sqrt{x^2 + y^2}$

Solution

We want to find the volume of the solid region E bounded above by the plane $z = 2$ and bounded below by the cone $z = \sqrt{x^2 + y^2}$.

The equations involve $x^2 + y^2$, and the solid has an axis of symmetry along the z-axis. This makes cylindrical coordinates (r, θ, z) the most convenient choice. The coordinate transformations are:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ x^2 + y^2 &= r^2 \end{aligned}$$

The volume element in cylindrical coordinates is $dV = r\, dz\, dr\, d\theta$. The equations of the bounding surfaces become:

- Cone: $z = \sqrt{r^2} \implies z = r$
- Plane: $z = 2$

The volume V is given by the triple integral over the region E:

$$V = \iiint_E dV$$

We need to determine the limits of integration for z, r, and θ.

Limits for z: The solid is bounded below by the cone ($z = r$) and above by the plane ($z = 2$). Thus, the limits for z are:

$$r \le z \le 2$$

Limits for r and θ: The limits for r and θ are determined by the projection of the solid onto the xy-plane. This projection is the region where the cone intersects the plane. We find this by setting the z-values equal:

$$r = 2$$

This describes a circle of radius 2 centered at the origin. Therefore, the radius r ranges from 0 to 2, and the angle θ ranges over a full circle from 0 to 2π.

$$\begin{aligned} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{aligned}$$

The Integral: Combining these limits, the volume integral is:

$$V = \int_0^{2\pi} \int_0^2 \int_r^2 r\,dz\,dr\,d\theta$$

We evaluate the integral from the inside out.

Step 1: Integrate with respect to z

$$\int_r^2 r\,dz = r\,[z]_r^2 = r(2-r) = 2r - r^2$$

Step 2: Integrate with respect to r Substitute the result from Step 1 into the next integral:

$$\int_0^2 (2r - r^2)\,dr = \left[r^2 - \frac{r^3}{3}\right]_0^2 = \left(2^2 - \frac{2^3}{3}\right) - (0) = 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$

Step 3: Integrate with respect to θ Substitute the result from Step 2 into the final integral:

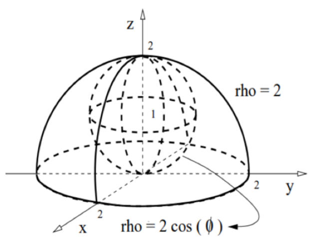
$$\int_0^{2\pi} \frac{4}{3}\,d\theta = \frac{4}{3}\,[\theta]_0^{2\pi} = \frac{4}{3}(2\pi - 0) = \frac{8\pi}{3}$$

The volume of the solid bounded by the plane $z = 2$ and the cone $z = \sqrt{x^2 + y^2}$ is:

$$V = \frac{8\pi}{3}$$

Question 10

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.



Solution

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos(\phi)}^2 \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta.$$

$$V = \int_0^{2\pi} \int_0^{\pi/2} \left(\frac{\rho^3}{3}\right)\Bigg|_{2\cos(\phi)}^2 \sin(\phi)\,d\phi\,d\theta$$

$$V = \int_0^{2\pi} \int_0^{\pi/2} \left[\frac{2^3}{3} - \frac{(2\cos(\phi))^3}{3}\right] \sin(\phi)\,d\phi\,d\theta$$

$$V = \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{3}\left[8 - 8\cos^3(\phi)\right] \sin(\phi)\,d\phi\,d\theta$$

$$V = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/2} \left[1 - \cos^3(\phi)\right] \sin(\phi)\,d\phi\,d\theta$$

$$V = \frac{8}{3}(2\pi) \int_0^{\pi/2} \left[\sin(\phi) - \cos^3(\phi)\sin(\phi)\right]\,d\phi$$

$$V = \frac{16\pi}{3}\left[(-\cos(\phi))\Bigg|_0^{\pi/2} - \int_0^{\pi/2} \cos^3(\phi)\sin(\phi)\,d\phi\right].$$

Introduce the substitution for the integral: $u = \cos(\phi)$, $du = -\sin(\phi)\, d\phi$.

When $\phi = 0$, $u = \cos(0) = 1$.
When $\phi = \pi/2$, $u = \cos(\pi/2) = 0$.

Now substitute into the integral:

$$\int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi = \int_1^0 u^3(-du) = -\int_1^0 u^3\, du = \int_0^1 u^3\, du.$$

So, the entire expression for V becomes:

$$\begin{aligned}
V &= \frac{16\pi}{3}\left[1 - \int_0^1 u^3\, du\right] \\
V &= \frac{16\pi}{3}\left[1 - \left(\frac{u^4}{4}\right)\Bigg|_0^1\right] \\
V &= \frac{16\pi}{3}\left[1 - \left(\frac{1^4}{4} - \frac{0^4}{4}\right)\right] \\
V &= \frac{16\pi}{3}\left[1 - \frac{1}{4}\right] \\
V &= \frac{16\pi}{3}\left(\frac{3}{4}\right) \\
V &= \frac{16\pi \cdot 3}{3 \cdot 4} \\
V &= 4\pi.
\end{aligned}$$

Question 11

Given a solid bound by $z = 2$ and $z = \sqrt{x^2 + y^2}$, find the mass density if the mass density is directly proportional to the square of the distance from origin.

Solution
The problem states that the mass density, let's call it $\delta(x, y, z)$, is directly proportional to the square of the distance from the origin. The square of the distance from the origin to a point (x, y, z) is $x^2 + y^2 + z^2$.

Therefore, the density function is:

$$\delta(x, y, z) = k(x^2 + y^2 + z^2)$$

where k is the constant of proportionality.

Given the geometry of the solid (bounded by a cone), it is best to use spherical coordinates (ρ, ϕ, θ). In spherical coordinates, the square of the distance from the origin is simply ρ^2. So, the density function becomes:

$$\delta(\rho, \phi, \theta) = k\rho^2$$

The solid is bounded by the plane $z = 2$ and the cone $z = \sqrt{x^2 + y^2}$. We must convert these boundaries into spherical coordinates.

The Cone: In cylindrical coordinates, the cone is $z = r$. Using the transformations $z = \rho\cos(\phi)$ and $r = \rho\sin(\phi)$, we get:

$$\rho\cos(\phi) = \rho\sin(\phi) \implies \tan(\phi) = 1 \implies \phi = \frac{\pi}{4}$$

This means the cone forms a constant angle with the positive z-axis. The solid is above this cone, so the angle ϕ ranges from the z-axis ($\phi = 0$) to the cone itself.

$$0 \leq \phi \leq \frac{\pi}{4}$$

The Plane: The upper bound is the plane $z = 2$. Using $z = \rho\cos(\phi)$:

$$\rho\cos(\phi) = 2 \implies \rho = \frac{2}{\cos(\phi)} \implies \rho = 2\sec(\phi)$$

The distance ρ starts from the origin ($\rho = 0$) and extends to this plane.

$$0 \leq \rho \leq 2\sec(\phi)$$

The Angle θ: Since the solid is symmetric about the z-axis, the angle θ completes a full revolution.

$$0 \leq \theta \leq 2\pi$$

The total mass M is the triple integral of the density function over the volume of the solid. The volume element in spherical coordinates is $dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$.

$$M = \iiint_E \delta\, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sec(\phi)} (k\rho^2)(\rho^2 \sin(\phi))\, d\rho\, d\phi\, d\theta$$

$$M = k \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sec(\phi)} \rho^4 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Step 1: Integrate with respect to ρ

$$\int_0^{2\sec(\phi)} \rho^4 \sin(\phi)\, d\rho = \sin(\phi) \left[\frac{\rho^5}{5}\right]_0^{2\sec(\phi)} = \sin(\phi)\frac{(2\sec(\phi))^5}{5} = \frac{32}{5}\frac{\sin(\phi)}{\cos^5(\phi)} = \frac{32}{5}\tan(\phi)\sec^4(\phi)$$

Step 2: Integrate with respect to ϕ

$$\int_0^{\pi/4} \frac{32}{5}\tan(\phi)\sec^4(\phi)\, d\phi$$

We use the substitution $u = \tan(\phi)$, so $du = \sec^2(\phi)\, d\phi$. We rewrite $\sec^4(\phi) = \sec^2(\phi)\sec^2(\phi) = (1 + \tan^2(\phi))\sec^2(\phi) = (1 + u^2)\sec^2(\phi)$.

The limits change: when $\phi = 0, u = 0$; when $\phi = \pi/4, u = 1$.

$$\begin{aligned}
\frac{32}{5}\int_0^1 u(1+u^2)\, du &= \frac{32}{5}\int_0^1 (u + u^3)\, du \\
&= \frac{32}{5}\left[\frac{u^2}{2} + \frac{u^4}{4}\right]_0^1 \\
&= \frac{32}{5}\left(\frac{1}{2} + \frac{1}{4} - 0\right) = \frac{32}{5}\left(\frac{3}{4}\right) = \frac{24}{5}
\end{aligned}$$

Step 3: Integrate with respect to θ

$$\int_0^{2\pi} k\left(\frac{24}{5}\right) d\theta = \frac{24k}{5}\left[\theta\right]_0^{2\pi} = \frac{24k}{5}(2\pi) = \frac{48k\pi}{5}$$

The total mass of the solid is:

$$M = \frac{48k\pi}{5}$$

where k is the constant of proportionality.

Question 12

Find the mass of T, $\rho(x, y, z) = y$, where T is region bound by $y = x^2 + z^2$ and $y = 4$

Solution
We need to find the mass M of a solid region T. The density is given by the function $\rho(x, y, z) = y$. The solid T is bounded by the surfaces:

- A paraboloid opening along the y-axis: $y = x^2 + z^2$
- A plane perpendicular to the y-axis: $y = 4$

The formula for mass is given by the triple integral of the density function over the region T:

$$M = \iiint_T \rho(x, y, z)\, dV$$

The solid has a circular cross-section in the xz-plane and is symmetric about the y-axis. This suggests using a modified version of cylindrical coordinates. Let's define our coordinates as follows:

$$\begin{aligned} x &= r\cos(\theta) \\ z &= r\sin(\theta) \\ y &= y \end{aligned}$$

In this system, $x^2 + z^2 = r^2$. The volume element is $dV = r\, dy\, dr\, d\theta$.
The bounding surfaces and the density function become:

- Paraboloid: $y = r^2$
- Plane: $y = 4$
- Density: $\rho(r, y, \theta) = y$

We set up the integral in the order $dy\, dr\, d\theta$.

Limits for y: For any point (r, θ) in the xz-plane, the vertical extent of the solid goes from the paraboloid surface up to the plane.

$$r^2 \leq y \leq 4$$

Limits for r and θ: The projection of the solid onto the xz-plane is a disk. The radius of this disk is found at the intersection of the two surfaces:

$$r^2 = 4 \implies r = 2$$

Thus, the radius r ranges from the y-axis ($r = 0$) to the edge of the disk ($r = 2$). Since the disk is complete, the angle θ ranges over a full circle.

$$\begin{aligned} 0 &\leq r \leq 2 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

Now we substitute the density, volume element, and limits into the mass formula.

$$M = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (y) \cdot (r\, dy\, dr\, d\theta)$$

We evaluate the integral from the inside out.

Step 1: Integrate with respect to y

$$\int_{r^2}^4 ry\, dy = r\left[\frac{y^2}{2}\right]_{r^2}^4 = r\left(\frac{4^2}{2} - \frac{(r^2)^2}{2}\right) = r\left(\frac{16}{2} - \frac{r^4}{2}\right) = 8r - \frac{1}{2}r^5$$

Step 2: Integrate with respect to r

$$\int_0^2 \left(8r - \frac{1}{2}r^5\right)\, dr = \left[4r^2 - \frac{r^6}{12}\right]_0^2 = \left(4(2^2) - \frac{2^6}{12}\right) - (0) = 16 - \frac{64}{12} = 16 - \frac{16}{3} = \frac{32}{3}$$

Step 3: Integrate with respect to θ

$$\int_0^{2\pi} \frac{32}{3}\, d\theta = \frac{32}{3}\, [\theta]_0^{2\pi} = \frac{32}{3}(2\pi - 0) = \frac{64\pi}{3}$$

The total mass of the solid is:

$$M = \frac{64\pi}{3}$$