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Tutorial 12: Line Integrals and Green's Theorem

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Question 1

If $A = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C A \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C:

a. $x = t, y = t^2, z = t^3$

b. The straight line from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$, and then to $(1,1,1)$.

c. The straight line joining $(0,0,0)$ and $(1,1,1)$.

Solution

a.

$$\begin{aligned} r(t) &= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \\ &= t\hat{i} + t^2\hat{j} + t^3\hat{k} \end{aligned}$$

$$\begin{aligned} A(r(t)) &= (3t^2 + 6t^2)\hat{i} - 14(t^2)(t^3)\hat{j} + 20(t)(t^3)^2\hat{k} \\ &= 9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k} \end{aligned}$$

$$\frac{dr}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\begin{aligned} \int_C F \cdot dr &= \int_0^1 F(r(t)) \cdot \frac{dr}{dt} \cdot dt \\ &= \int_0^1 (9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k})(\hat{i} + 2t\hat{j} + 3t^2\hat{k})dt \\ &= \int_0^1 9t^2 - 28t^6 + 60t^9 dt \\ &= 3t^3 - 4t^7 + 6t^10\Big|_0^1 \\ &= 3 - 4 + 6 \\ &= 5 \end{aligned}$$

b. From $(0,0,0)$ to $(1,0,0)$,

$$r = (0,0,0) + t(1,0,0) = t\hat{i} + 0\hat{j} + 0\hat{k} \Rightarrow x = t, y = 0, z = 0$$

$$\frac{dr}{dt} = 1\hat{i}$$

$$\begin{aligned} \int_0^1 F(r(t)) \cdot \frac{dr}{dt} \cdot dt &= \int_0^1 (3t^2\hat{i}) \cdot (\hat{i})dt \\ &= t^3\Big|_0^1 = 1 \end{aligned}$$

From $(1,0,0)$ to $(1,1,0)$,

$$r = (1,0,0) + t(0,1,0) = \hat{\imath} + t\hat{\jmath} + 0\hat{k} \Rightarrow x = 1, y = t, z = 0$$

$$\frac{dr}{dt} = 1\hat{\jmath}$$

$$F(r,t) = 3(1) + 6t\hat{\imath} - 14(t)(0)\hat{\jmath} + 20(1)(0)\hat{k} = 3 + 6t\hat{\imath}$$

$$\begin{aligned}\int_0^1 F(r(t)) \cdot \frac{dr}{dt} \cdot dt &= \int_0^1 (3 + 6t\hat{\imath}) \cdot (\hat{\jmath})dt \\ &= 0\end{aligned}$$

From $(1,1,0)$ to $(1,1,1)$,

$$r = (1,1,0) + t(0,0,1) = \hat{\imath} + \hat{\jmath} + t\hat{k} \Rightarrow x = 1, y = 1, z = t$$

$$\frac{dr}{dt} = 1\hat{k}$$

$$\begin{aligned}F(r,t) &= (3(1) + 6(1))\hat{\imath} - 14(1)(t)\hat{\jmath} + 20(1)(t)^2\hat{k} \\ &= 9\hat{\imath} - 14t\hat{\jmath} + 20t^2\hat{k}\end{aligned}$$

$$\begin{aligned}\int_0^1 F(r(t)) \cdot \frac{dr}{dt} \cdot dt &= \int_0^1 (9\hat{\imath} - 14t\hat{\jmath} + 20t^2\hat{k}) \cdot (\hat{k})dt \\ &= \int_0^1 20t^2 dt \\ &= \frac{20}{3}t^3\Big|_0^1 = \frac{20}{3}\end{aligned}$$

$\therefore$ Total Line Integral $= 1 + 0 + \frac{20}{3} = \underline{\underline{\frac{23}{3}}}$

c.

$$r = t\hat{\imath} + t\hat{\jmath} + t\hat{k} \Rightarrow x = t, y = t, z = t$$

$$\frac{dr}{dt} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$\begin{aligned}&\int_0^1 \left[(3t^2 + 6t)\hat{\imath} - 14(t)(t)\hat{\jmath} + 20(t)(t)^2\hat{k}\right] \cdot \left[\hat{\imath} + \hat{\jmath} + \hat{k}\right] \\ &= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3)dt \\ &= \int_0^1 (6t - 11t^2 + 20t^3)dt \\ &= 3t^2 - \frac{11}{3}t^3 + 5t^4\Big|_0^1 \\ &= \frac{13}{3}\end{aligned}$$

Question 2

Find the work done in moving a particle in a force field given by $F = 3xy\hat{\imath} - 5z\hat{\jmath} + 10x\hat{k}$ along the curve $x = t^2+1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.

Solution

$$r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} = (t^2+1)\hat{\imath} + 2t^2\hat{\jmath} + t^3\hat{k}$$

$$\frac{dr}{dt} = 2t\hat{\imath} + 4t\hat{\jmath} + 3t^2\hat{k}$$

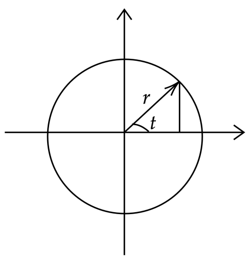
$$F(r(t)) = 3(t^2+1)(2t^2)\hat{\imath} - 5(t^3)\hat{\jmath} + 10(t^2+1)\hat{k}$$

$$\begin{aligned}\int_C F \cdot dr &= \int_1^2 (6t^4+6t^2)\hat{\imath} - 5t^3\hat{\jmath} + (10t^2+10)\hat{k} \cdot (2t\hat{\imath} + 4t\hat{\jmath} + 3t^2\hat{k})dt \\ &= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)dt \\ &= 2t^6 + 3t^4 - 4t^5 + 6t^5 + 10t^3\Big|_1^3 \\ &= 2t^6 + 2t^5 + 3t^4 + 10t^3\Big|_1^2 \\ &= (128+64+48+80) - (2+2+3+10) \\ &= 320 - 17 = 303\end{aligned}$$

Question 3

Determine the work done in moving a particle once around a circle C in the xy-plane, if the circle has center at the origin and radius of 3 and if those field is given by $F = (2x-y+z)\hat{\imath} + (x+y-z^2)\hat{\jmath} + (3x-2y+4z)\hat{k}$.

Solution



$$r = x\hat{\imath} + y\hat{\jmath} = 3\cos t\hat{\imath} + 3\sin t\hat{\jmath}$$

In the plane $z = 0$,

$$\mathbf{F} = (2x-y)\hat{\imath} + (x+y)\hat{\jmath} + (3x-2y)\hat{k}$$

$$dr = dx\hat{\imath} + dy\hat{\jmath}$$

The work done is,

$$\begin{aligned}\int_C \mathbf{F} \cdot dt &= \int_C \left[(2x-y)\hat{\imath} + (x+y)\hat{\jmath} + (3x-2y)\hat{k}\right] \cdot [dx\hat{\imath} + dy\hat{\jmath}] \\ &= \int_C (2x-y)dx + (x+y)dy\end{aligned}$$

Parametric equation of the circle: $x = 3\cos t$, $y = 3\sin t$, where t varies from 0 to 2π. $\Rightarrow dx = -3\sin t dt, dy = 3\cos t dt$

The line integral,

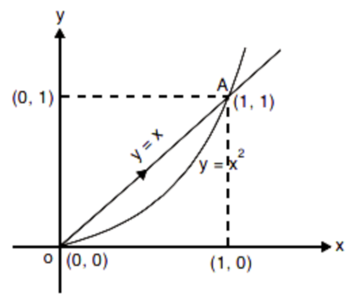
$$\int_{t=0}^{2\pi} [2(3\cos t - 3\sin t)(-3\sin t)dt] + [3\cos t + 3\sin t][3\cos t dt]$$
$$= \int_0^{2\pi} -18\sin t\cos t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t dt$$
$$= \int_0^{2\pi} (9 - 9\sin t\cos t)dt$$
$$= 9t - \frac{9}{2}\sin^2 t\Big|_0^{2\pi}$$
$$= 18\pi$$

Note: We choose counterclockwise direction, which is known as positive direction. If traversing from counter-clockwise (negative) direction, the value of integral would be -18π.

Question 4

Find the line integral for $\oint_C (xy + y^2)\, dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$ in the positive direction in traversing C. $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$.

Solution



Along $y = x^2$, where $dy = 2xdx$ the line integral equals

$$\int_0^1 \left((x)(x^2) + (x^2)^2\right) dx + (x^2)(2x)dx = \int_0^1 (3x^3 + x^4)dx = \frac{19}{20}$$

Along $y = x$ from $(1,1)$ to $(0,0)$, the line integral equals

$$\int_0^1 \left((x)(x) + (x)^2\right) dx + x^2 dx = \int_0^1 3x^2 dx = 1$$

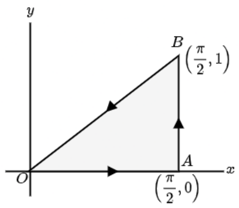
L.H.S = The required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dxdy$$
$$= \iint_R (x - 2y)dxdy$$
$$= \int_{x=0}^1 \int_{y=x^2}^x (x - 2y)dydx$$
$$= \int_0^1 \left[\int_{x^2}^x (x - 2y)dy\right] dx$$
$$= \int_0^1 (xy - y^2)\big|_{x^2}^x\, dx$$
$$= \int_0^1 \left[(x^2 - x^2) - (x^3 - x^4)\right] dx$$
$$= \int_0^1 \left(x^4 - x^3\right) dx$$
$$= \frac{x^5}{5} - \frac{x^4}{4}\Big|_0^1$$
$$= \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

L.H.S = R.H.S $= -\frac{1}{20}$.

Question 5

Evaluate $\oint_C (y - \sin x)dx + \cos x dy$ where C is the triangle of the adjoining figure by using Green's theorem in the plane.



Solution

$M = y - \sin x,\ N = \cos x,\ \dfrac{\partial N}{\partial x} = -\sin x,\ \dfrac{\partial M}{\partial y} = 1,$

$$\begin{aligned}
\oint_C M dx + N dy &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \\
&= \iint_R (-\sin x - 1) dydx \\
&= \int_{x=0}^{\frac{\pi}{2}} \left[\int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)dy\right] dx \\
&= \int_{x=0}^{\frac{\pi}{2}} (-y\sin x - y)\Big|_0^{\frac{2x}{\pi}} dx \\
&= \int_0^{\frac{\pi}{2}} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx \\
&= -\frac{2}{\pi}(-x\cos x + \sin x) - \frac{x^2}{\pi}\Big|_0^{\frac{\pi}{2}} \\
&= -\frac{2}{\pi} - \frac{\pi}{4}
\end{aligned}$$

Note: Although there exist lines parallel to the coordinate axes (coincident with the coordinate axes in this case) which meet C in an infinite number of points, Green's theorem in the plane still holds. In general the theorem is valid when C is composed of a finite number of straight line segments.

Question 6

Calculate $\oint_C -x^2 y dx + xy^2 dy$ where C is the circle of radius 2 centered on the origin.

Solution

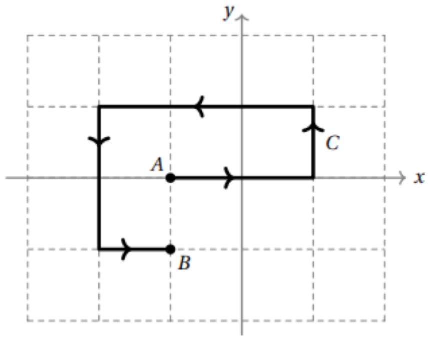
$$\oint_C M dx + N dy = \iint_R N_x - M_y dA$$

We let $M = -x^2 y$ and $N = xy^2$ to get

$$\begin{aligned}
\oint_C -x^2 y dx + xy^2 dy &= \iint_R y^2 - (-x^2) dA \\
&= \iint_R x^2 + y^2 dA \\
&= \int_0^{2\pi} \int_0^2 r^2 r dr d\theta \\
&= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta \\
&= \int_0^{2\pi} 4 d\theta \\
&= 4\theta\Big|_0^{2\pi} = 8\pi
\end{aligned}$$

Question 7

ompute the line integral of $\mathbf{F}(x, y) = \langle x^3, 4x \rangle$ along the path C shown below against a grid of unit-sized squares. Use Green's theorem to relate this to a line integral over the vertical path joining B to A.



Solution

Let L be the line segment going from B to A. Then, we can now apply Green's theorem to combination of C and L. Let D be the region bounded by these two paths. Then, by Green's theorem, since we are oriented correctly,

$$\begin{aligned}\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \\ &= \iint_D \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(x^3) dA \\ &= \iint_D 4dA \\ &= 4 \cdot \text{Area}(A) \\ &= 16\end{aligned}$$

because the area of the region is made of exactly 4 unit squares. The boundary of D is C and L:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_L \mathbf{F} \cdot d\mathbf{r} = 16$$

The line integral along L is easier: parametrizing L by $\mathbf{r}(t) = (-1, t)$ for $-1 \leq t \leq 0$, we get

$$\int_{\partial L} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{0} (-1, -4) \cdot (0, 1) dt = \int_{-1}^{0} -4dt = -4$$

Putting it together,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 16 - \int_L \mathbf{F} \cdot d\mathbf{r} = 16 - (-4) = 20$$

Question 8

A particle starts at $(-2,0)$ and moves along the x-axis to $(2,0)$. Then it moves along the upper part of the circle $x^2+y^2=4$ and back to $(-2,0)$. Compute the work done on this particle by the force field $\boldsymbol{F}(x,y)=\left\langle x, x^3+3xy^2\right\rangle$.

Solution

Let $\mathbf{F}(x,y)=P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$ with $P(x,y)=\sin(x^3)$ and $Q(x,y)=2ye^{x^2}$.

By Green's theorem,

$$\begin{aligned}
\int_C Pdx+Qdy &= \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy \\
&= \int_0^2\int_0^y 4xye^{x^2}dxdy \\
&= \int_0^2 2y\left(e^{y^2}-1\right)dy \\
&= e^4-5
\end{aligned}$$

Question 9

Let $\mathbf{F}(x,y)=\langle \sin x, \cos y\rangle$ and let C be the curve that is the top half of the circle $x^2+y^2=1$, traversed counterclockwise from $(1,0)$ to $(-1,0)$, and the line segment from $(-1,0)$ to $(-2,3)$. Evaluate the line integral $\int \mathbf{F}\cdot\mathbf{T}ds=\int_C \sin x dx+\cos y dy$

Solution

We consider the integrals over the semicircle, denoted by C_1, and the line segment, denoted by C_2, seperately. We then have,

$$\int_C \sin x dx+\cos y dy=\int_{C_1}\sin x dx+\cos y dy+\int_{C_2}\sin x dx+\cos y dy$$

For the semicircle, we use the parametric equations

$$x=\cos t,\quad y=\sin t,\quad 0\le t\le\pi$$

This yields

$$\begin{aligned}
\int_{C_1}\sin x dx+\cos y dy &= \int_0^\pi \sin(\cos t)(-\sin t)(\cos t)dt+\cos(\sin t)dt \\
&= -\cos(\cos t)|_0^\pi+\sin(\sin t)|_0^\pi \\
&= -\cos(-1)+\cos(1) \\
&= 0
\end{aligned}$$

For the line segment, we use the parametric equations

$$x=-1-t,\quad y=3t,\quad 0\le t\le 1$$

This yields

$$\begin{aligned}
\int_{C_2}\sin x dx+\cos y dy &= \int_0^1 \sin(-1-t)(-1)dt+\cos(3t)(3)dt \\
&= -\cos(-1-t)|_0^1+\sin(3t)|_0^1 \\
&= -\cos(-2)+\cos(-1)+\sin(3)-\sin(0) \\
&= -\cos(2)+\cos(1)+\sin(3)
\end{aligned}$$

We conclude

$$\int_C \sin x dx + \cos y dy = \cos(1) - \cos(2) + \sin(3)$$

In evaluating these integrals, we have taken advantage of the rule,

$$\int_a^b f'(g(t))g'(t)dt = f(g(b)) - f(g(a))$$