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Tutorial 13: Surface Integrals

Prepared by: Hong Vin | Website: `https://kix1001.hongvin.xyz`

Question 1

Evaluate the surface integral of the vector field $\mathbf{F} = 3x^2\mathbf{i} - 2yx\mathbf{j} + 8\mathbf{k}$ over the surface S that is the graph of $z = 2x - y$ over the rectangle $[0, 2] \times [0, 2]$.

Solution
Use the formula for a surface integral over a graph $z = g(x, y)$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[\mathbf{F} \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)\right] dxdy$$

For $z = 2x - y \to -2x + y + z = 0$, therefore,

$$\begin{aligned}\int_0^2 \int_0^2 (3x^2, -2yx, 8) \cdot (-2, 1, 1) dxdy &= \int_0^2 \int_0^2 (-6x^2 - 2yx + 8) dxdy \\ &= \int_0^2 -2x^3 - yx^2 + 8x\Big|_{x=0}^2 dy \\ &= \int_0^2 -4ydy \\ &= -2y^2\Big|_0^2 = -8\end{aligned}$$

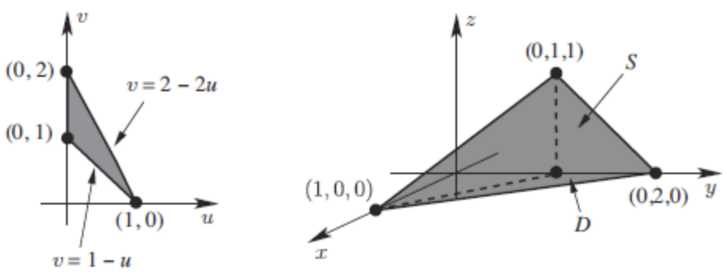
Question 2

Let S be the triangle with vertices $(1, 0, 0), (0, 2, 0)$ and $(0, 1, 1)$ and let $\mathbf{F} = xyz(\mathbf{i} + \mathbf{j})$. calculate the surface integral

$$\iint_S F \cdot d\mathbf{S}$$

If the triangle is oriented by the "downward" normal.

Solution
Since S lies in a plane, it is part of the graph of a linear function $z = ax + by + c$.



Substituting the vertices of the triangle for (x, y, z), we get the equation

$$0 = a + c, \quad 0 = 2b + c, \quad 1 = b + c$$

which we can solve to find $b = -1$, $= 2$, $a = -2$, i.e., $z = -2x - y + 2$. We may take x and y as parameters,

$$x = u, \quad y = v, \quad z = -2u - v + 2$$

or $\mathbf{\Phi}(u, v) = (u, v, -2u - v + 2)$. The domain D of the parametrization is the triangle with vertices at $(1, 0)$, $(0, 2)$, and $(0, 1)$ in the (u, v) plane. For this parametrization,

$$\mathbf{T}_u \times \mathbf{T}_v = (1, 0, -2) \times (0, 1, -1) = (2, 1, 1)$$

Since the third component of this vector is positive, the orientation determined by $\mathbf{\Phi}$ is "upward", so we will have to multiply our find answer by -1 to get the surface integral with the **downward** orientation.

Now, we have (with the minus sign reminding us that the orientation is wrong),

$$\begin{aligned} -\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D xyz(\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} + \mathbf{j} + \mathbf{k}) du dv \\ &= \iint_D 3xyz du dv \\ &= \iint_D 3uv(-2u - v + 2) du dv \end{aligned}$$

To compute the double integral, we draw the integtation domain D in the uv-plane, in the left hand part of the Figure. By reduction to iterated integrals,

$$\iint_D 3uv(-2u - v + 2) du dv = \int_0^1 \int_{1-u}^{2-2u} (-6u^2 v - 3uv^2 + 6uv) dv du$$

Carrying out the v-integration, we get

$$\begin{aligned} &\int_0^1 \left[-3u^2 v^2 - uv^3 + 3uv^2\right]\Big|_{1-u}^{2-2u} du \\ &= \int_0^1 uv^2[-3u - v + 3]\Big|_{1-u}^{2(1-u)} du \\ &= \int_0^1 \left[4u(1-u)^2(1-u) - u(1-u)^2 2(1-u)\right] du \\ &= 2\int_0^1 u(1-u)^3 du \\ &= 2\int_0^1 (u - 3u^2 + 3u^3 - u^4) du \\ &= 2\left(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right) = \frac{1}{10} \end{aligned}$$

$$\therefore \iint_S \mathbf{F} \cdot d\mathbf{S} = -\frac{1}{10}$$

Question 3

> The equations $z = 12, x^2 + y^2 \leq 25$ describe a disk of radius 5 lying in the plane $z = 12$. Suppose that is the position vector field $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute $\iint_S \mathbf{r}.d\mathbf{S}$.

Solution

Since the disk is parallel to the xy plane, the outward unit normal is $\mathbf{k}$. Hence $\mathbf{n}(x, y, z) = \mathbf{k}$ and so $\mathbf{r} \cdot \mathbf{n} = z$. Thus,

$$\iint_S \mathbf{r} \cdot d\mathbf{S} = \iint_S \mathbf{r} \cdot \mathbf{n} dS = \iint_S z dS = \iint_D 12 dx dy = \underline{\underline{300\pi}}$$

Alternatively we may solve this problem by using the formula for surface integrals over graphs:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right) dx dy$$

With $g(x, y) = 12$ and D the disk $x^2 + y^2 \leq 25$, we get

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (x \cdot 0 + y \cdot 0 + 12) dx dy = 12(\text{area of } D) = 300\pi$$

Question 4

> Let S be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1 \geq 0$, and its base $x^2 + y^2 \leq 1, z = 0$. Let E be the electric field defined by $\mathbf{E}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Find the electric flux across S.

Solution

Write $S = H \cup D$ where H is the upper hemisphere and D is the disk. Hence

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_H \mathbf{E} \cdot d\mathbf{S} + \iint_D \mathbf{E} \cdot d\mathbf{S}$$

(i) Let $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the unit normal n pointing outward from H. Then

$$\begin{aligned} \iint_H \mathbf{E} \cdot d\mathbf{S} &= \iint_H \mathbf{E} \cdot \mathbf{n} dS \\ &= \iint_H (2x, 2y, 2z) \cdot (x, y, z) dS \\ &= 2 \iint_H (x^2 + y^2 + z^2) dS \\ &= 2 \iint_H dS = 4\pi \end{aligned}$$

(ii) The unit normal is $-\mathbf{k}$ and $z = 0$ on D. Hence,

$$\iint_D \mathbf{E} \cdot d\mathbf{S} = \iint_D \mathbf{E} \cdot \mathbf{n} dS = \iint_D (2x, 2y, 2z) \cdot (0, 0, -1) dS = 0$$

Therefore,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi$$

Question 5

Find the area of the ellipse cut on the plane $2x + 3y + 6z = 60$ by the circular cylinder $x^2 + y^2 = 2x$.

Solution

The surface S lies in the plane $2x + 3y + 6z = 60$ so we use this to calculate $dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$. Differentiating the equation for the plane with respect to x gives,

$$2 + 6\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{1}{3}$$

Differentiating the equation for the plane with respect to y gives,

$$3 + 6\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial y} = -\frac{1}{2}$$

Hence,

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{1}{9} + \frac{1}{4}} = \frac{7}{6}$$

Then the area of S is found be calculating the surface integral over S for the function $f(x, y, z) = 1$. The projection of the surface, S, onto the xy-plane is given by $D = \{(x, y) : x^2 - 2x + y^2 = (x - 1)^2 + y^2 \leq 1\}$. Hence the area of S is given by

$$\begin{aligned}
\iint_S 1dS &= \iint_D 1 \cdot \frac{7}{6} dxdy \\
&= \frac{7}{6} \iint_D 1dxdy \\
&= \frac{7}{6} \times \text{Area of } D \\
&= \frac{7}{6}\pi
\end{aligned}$$

Note, since D is a circle or radius 1 centred at $(1, 0)$ the area of D is the area of a unit circle which is π.

Question 6

Find the integral $\iint_S xdS$, where the surface S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant.

Solution

It is convenient to solve this integral in spherical coordinates. The area element for spherical surface is $dS = a^2 \sin\theta d\phi d\theta$. As $x = a\cos\phi\sin\theta$, we can write the integral in the following form

$$\begin{aligned}
I = \iint_S xdS &= \iint_{D(\phi,\theta)} a\cos\phi\sin\theta \cdot a^2 \sin\theta d\phi d\theta \\
&= a^3 \iint_{D(\phi,\theta)} \cos\phi \sin^2\theta d\phi d\theta
\end{aligned}$$

The domain of integration $D(\phi, \theta)$ is defined as

$$D = \left\{(\phi, \theta)| 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}\right\}$$

Hence, the integral is

$$\begin{aligned}
I &= a^3 \iint_{D(\phi,\theta)} \cos\phi \sin^2\theta d\phi d\theta \\
&= a^3 \int_0^{\frac{\pi}{2}} \cos\phi d\phi \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta \\
&= a^3 \cdot \left[(\sin\phi)|_0^{\frac{\pi}{2}} \right] \cdot \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} d\theta \\
&= a^3 \cdot 1 \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} (1-\cos 2\theta) d\theta \\
&= \frac{a^3}{2} \left[\left(\theta - \frac{\sin 2\theta}{2} \right) \Big|_0^{\frac{\pi}{2}} \right] \\
&= \frac{a^3}{2} \cdot \frac{\pi}{2} = \frac{\pi a^3}{4}
\end{aligned}$$

Question 7

Find the integral $\iint_S \frac{dS}{\sqrt{x^2+y^2+z^2}}$, where S is the part of the cylindrical surface parameterized by $r(u,v) = (a\cos u, a\sin u, v), 0 \le u \le 2\pi, 0 \le v \le H$.

Solution
Calculate the partial derivatives,

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (-a\sin u, a\cos u, 0) \\
\frac{\partial \mathbf{r}}{\partial v} &= \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (0, 0, 1)
\end{aligned}$$

and their cross product,

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a\cos u \cdot \mathbf{i} + a\sin u \cdot \mathbf{j} + 0 \cdot \mathbf{k}$$

Then the area element of the given surface is

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv = \left| \sqrt{(a\cos u)^2 + (a\sin u)^2} \right| dudv = adudv$$

Now we can calculate the surface integral:

$$\begin{aligned}
\iint_S \frac{dS}{\sqrt{x^2+y^2+z^2}} &= \iint_{D(u,v)} \frac{adudv}{\sqrt{(a\cos u)^2 + (a\sin u)^2 + v^2}} \\
&= \iint_{D(u,v)} \frac{adudv}{\sqrt{a^2+v^2}} \\
&= \int_0^{2\pi} adu \int_0^H \frac{dv}{\sqrt{a^2+v^2}} \\
&= 2\pi a \int_0^H \frac{dv}{\sqrt{a^2+v^2}} \\
&= 2\pi a \left[\ln \left(v + \sqrt{a^2+v^2} \right) \Big|_{v=0}^H \right] = 2\pi a
\end{aligned}$$

$$\left[\ln \left(H + \sqrt{a^2+H^2} \right) - \ln a \right] = 2\pi a \ln \frac{H + \sqrt{a^2+H^2}}{a}$$