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Tutorial 14: Stokes' Theorem

Prepared by: Hong Vin | Website: `https://kix1001.hongvin.xyz`

Question 1

Suppose $\mathbf{F} = \langle -y, x, z\rangle$ and S is the part of the sphere $x^2+y^2+z^2 = 25$ below the plane z $= 4$, oriented with the outward-pointing normal (so that the normal at $(5,0,0)$ is $\mathbf{i}$). Compute the flux integral $\iint_S$ curl $\mathbf{F}.\mathrm{d}\mathbf{S}$ using Stokes' theorem.

Solution

Again we integrate the line integral over the boundary curve C rather than the flux integral over the (more complicated) surface S.

The boundary curve is the circle $x^2 + y^2 + 4^2 = 25$ (or $x^2 + y^2 = 9$) in the plane $z = 4$, but a note of caution is in order.

The natural parameterization (or the one we usually think of) is $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4\rangle$ actually parameterizes $-C$ (that is, C with the opposite orientation)!

Why is that? Imagine a person walking this boundary with their head in the normal (outward) direction. The remaining part of the sphere is on their right if they're walking counter-clockwise. It should be on their left, so they should be walking clockwise.

We'll calculate $\oint_{-C} \mathbf{F}\cdot d\mathbf{r}$ anyway, since we like the parameterisation. In terms of this parametrisation,

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) &= \langle -3\sin(t), 3\cos(t), 4\rangle \\
d\mathbf{r}(t) &= \langle -3\sin(t), 3\cos(t), 0\rangle dt \\
\mathbf{F}(\mathbf{r}(t))\cdot d\mathbf{r}(t) &= 9dt
\end{aligned}$$

Thus,

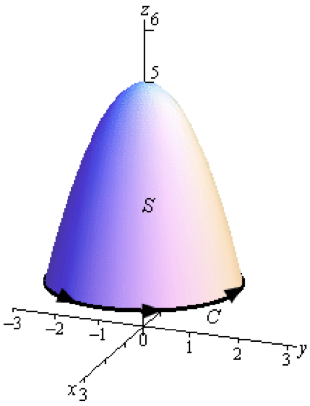
$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} 9dt = 18\pi$$

and so, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -18\pi.$

Question 2

Use Stokes' theorem to evaluate $\iint_S cur \mid \mathbf{F}.d\mathbf{S}$ where $F = z^2\mathbf{i} - 3xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the part of $z = 5 - x^2 - y^2$ above the $z = 1$. Assume that S is oriented upwards.

Solution



In this case the boundary curve C will be where the surface intersects the plane $z = 1$ and so will be the curve

$$\begin{gathered} 1 = 5 - x^2 - y^2 \\ x^2 + y^2 = 4 \quad \text{at } z = 1 \end{gathered}$$

So, the boundary curve will be the circle of radius 2 that is in the plane $z = 1$. The parameterization of this curve is,

$$\vec{r}\,(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + \vec{k}, \quad 0 \le t \le 2\pi$$

The first two components give the circle and the third component makes sure that it is in the plane $z = 1$.

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint\limits_S \text{curl}\vec{F} \cdot d\vec{S} = \int\limits_C \vec{F} \cdot d\,\vec{r} = \int_0^{2\pi} \vec{F}\,(\vec{r}\,(t)) \cdot \vec{r}\,'\,(t)\ dt$$

So, it looks like we need a couple of quantities before we do this integral. Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field.

$$\begin{aligned} \vec{F}\,(\vec{r}\,(t)) &= (1)^2\,\vec{i} - 3\,(2\cos t)\,(2\sin t)\ \vec{j} + (2\cos t)^3(2\sin t)^3\,\vec{k} \\ &= \vec{i} - 12\cos t\sin t\,\vec{j} + 64\cos^3 t\sin^3 t\,\vec{k} \end{aligned}$$

Next, we need the derivative of the parameterization and the dot product of this and the vector field.

$$\begin{aligned} \vec{r}\,'\,(t) &= -2\sin t\,\vec{i} + 2\cos t\,\vec{j} \\ \vec{F}\,(\vec{r}\,(t)) \cdot \vec{r}\,'\,(t) &= -2\sin t - 24\sin t\cos^2 t \end{aligned}$$

We can now do the integral.

$$\begin{aligned} \iint\limits_S \text{curl}\vec{F} \cdot d\vec{S} &= \int_0^{2\pi} -2\sin t - 24\sin t\cos^2 t\,dt \\ &= \left(2\cos t + 8\cos^3 t\right)\Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

Question 3

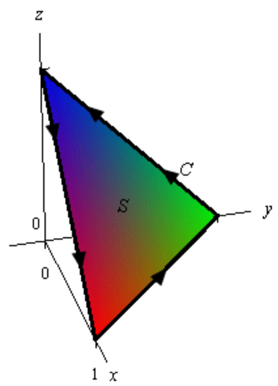
Use Stokes' theorem to evaluate $\int_c \mathbf{F}.d\mathbf{r}$ where $\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and C is the triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$ with counter clockwise rotation.

Solution

We are going to need the curl of the vector field eventually so let's get that out of the way first.

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 2z\,\vec{j} - \vec{j} = (2z-1)\,\vec{j}$$

Now, all we have is the boundary curve for the surface that we'll need to use in the surface integral. However, as noted above all we need is any surface that has this as its boundary curve. So, let's use the following plane with upwards orientation for the surface.



Since the plane is oriented upwards this induces the positive direction on C as shown. The equation of this plane is,

$$x + y + z = 1 \qquad \Rightarrow \qquad z = g\left(x,y\right) = 1 - x - y$$

Now, let's use Stokes' Theorem and get the surface integral set up.

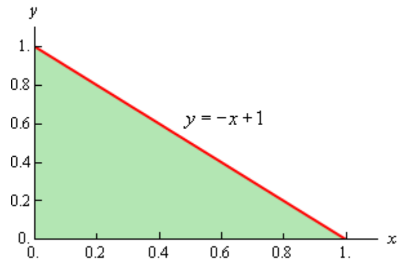
$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl}\vec{F} \cdot d\vec{S} \\ &= \iint_S (2z-1)\,\vec{j} \cdot d\vec{S} \\ &= \iint_D (2z-1)\,\vec{j} \cdot \frac{\nabla f}{\|\nabla f\|}\ \|\nabla f\|\, dA \end{aligned}$$

Okay, we now need to find a couple of quantities. First let's get the gradient. Recall that this comes from the function of the surface.

$$\begin{aligned} f\left(x,y,z\right) &= z - g\left(x,y\right) = z - 1 + x + y \\ \nabla f &= \vec{i} + \vec{j} + \vec{k} \end{aligned}$$

Note as well that this also points upwards and so we have the correct direction.

Now, D is the region in the xy-plane shown below,



We get the equation of the line by plugging in $z = 0$ into the equation of the plane. So based on this the ranges that define D are,

$$0 \le x \le 1 \qquad 0 \le y \le -x + 1$$

The integral is then,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \iint_D (2z-1)\,\vec{j} \cdot \left(\vec{i} + \vec{j} + \vec{k}\right) dA \\ &= \int_0^1 \int_0^{-x+1} 2(1-x-y) - 1\,dy\,dx\end{aligned}$$

Don't forget to plug in for z since we are doing the surface integral on the plane. Finishing this out gives,

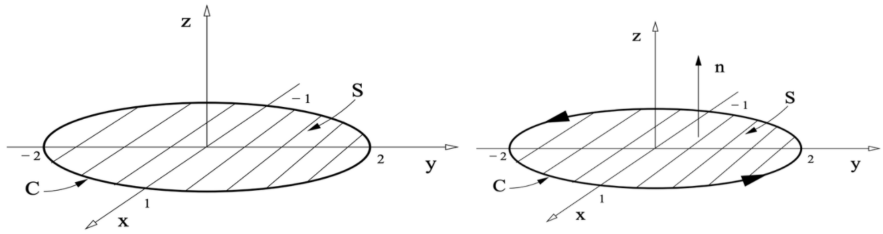
$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \int_0^{-x+1} 1 - 2x - 2y\,dy\,dx \\ &= \int_0^1 \left(y - 2xy - y^2\right)\Big|_0^{-x+1} dx \\ &= \int_0^1 x^2 - x\,dx \\ &= \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_0^1 \\ &= -\frac{1}{6}\end{aligned}$$

Question 4

Verify Stokes' Theorem for the field $\boldsymbol{F} = \langle x^2, 2x, z^2 \rangle$ on the ellipse $S = \left\{(x,y,z) : 4x^2 + y^2 \le 4, z = 0\right\}$

Solution

We compute both sides in $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$



We start computing the circulation integral on the ellipse $x^2 + \frac{y^2}{2^2} = 1$. We need to choose a counterclockwise parametrization, hence the normal to S points upwards.

We choose, for $t \in [0, 2\pi]$,

$$r(t) = \langle \cos(t), 2\sin(t), 0 \rangle$$

Therefore, the right-hand rule normal $\mathbf{n}$ to S is $\mathbf{n} = \langle 0, 0, 1 \rangle$

The circulation integral is:

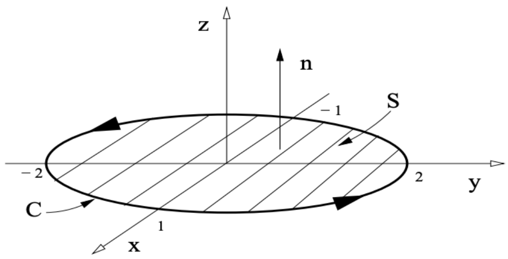
$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t) dt \\
&= \int_0^{2\pi} \langle \cos^2(t), 2\cos(t), 0 \rangle \cdot \langle -\sin(t), 2\cos(t), 0 \rangle dt \\
&= \int_0^{2\pi} [-\cos^2(t)\sin(t) + 4\cos^2(t)] dt
\end{aligned}$$

The substitution on the first term $u = \cos(t)$ and $du = -\sin(t)dt$, implies $\int_0^{2\pi} -\cos^2(t)\sin(t)dt = \int_1^1 u^2 du = 0$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 4\cos^2(t) dt = \int_0^{2\pi} 2[1 + \cos(2t)] dt$$

Since $\displaystyle\int_0^{2\pi} \cos(2t) dt = 0$, we conclude that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$

We now compute the right-hand side in Stokes' Theorem.



$$I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2x & z^2 \end{vmatrix} \quad \Rightarrow \quad \nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle$$

S is the flat surface $\{x^2 + \frac{y^2}{2^2} \leq 1, \quad z = 0\}$, so $d\sigma = dxdy$

Then, $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \langle 0, 0, 2 \rangle \cdot \langle 0, 0, 1 \rangle dy dx$

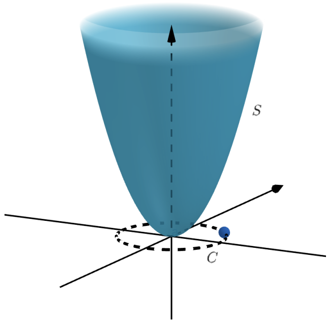
The right-hand side above is twice the area of the ellipse, Since that an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has area πab, we obtain

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = 4\pi$$

Question 5

Verify Stokes' Theorem for $\boldsymbol{F} = \left\langle y^2, -x, 5z \right\rangle$ and S is the paraboloid $z = x^2+y^2$ with the circle $x^2+y^2 = 1$ as its boundary.

Solution



Surface integral,

$$\nabla \times \vec{F} = \text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -x & 5z \end{vmatrix}$$
$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(-1-2y)$$
$$= (-1-2y)\hat{k}$$

$$\vec{n} = \langle f_x, f_y, 1 \rangle \quad \text{or} \quad \langle -f_x, -f_y, 1 \rangle$$
$$= \langle -2x, -2y, 1 \rangle$$

$$(\nabla \times \vec{F}) \cdot \vec{n} = 0(-2x) + 0(-2y) + (-1-2y)(1) = -1-2y$$

$$\iint_R (\nabla \times \vec{F}) \cdot \vec{n} dxdy = \iint_R (-1-2y) dxdy$$

R is region in side C, a unit circle.

Switching to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r \in [0,1], \quad \theta \in [0, 2\pi]$$

Hence,

$$= \int_0^{2\pi} \int_0^1 (-1 - 2r\sin\theta) r dr d\theta$$
$$= \int_0^{2\pi} \left(-\frac{r^2}{2} - 2\sin\theta \left(\frac{r^3}{3} \right) \right) \Bigg|_0^1 d\theta$$
$$= \int_0^{2\pi} \left(-\frac{1}{2} - \frac{2}{3}\sin\theta \right) d\theta$$
$$= \left(-\frac{1}{2}\theta + \frac{2}{3}\cos\theta \right) \Bigg|_0^{2\pi}$$
$$= -\frac{1}{2}(2\pi - 0) + \frac{2}{3}(\cos(2\pi) - \cos 0)$$
$$= -\pi$$

Line integral,

$$\begin{aligned}\oint_C \vec{F}\cdot\vec{T}ds &= \int_C Mdx + Ndy + Pdz \\ &= \int_C y^2dx - xdy + 5zdz\end{aligned}$$

C: unit circle $\rightarrow$ Switch to polar coordinates,

$$\begin{aligned}x &= \cos\theta, & y &= \sin\theta, & z &= 1 \\ dx &= -\sin\theta d\theta, & dy &= \cos\theta d\theta, & dz &= 0\end{aligned}$$

Hence,

$$\begin{aligned}&= \int_0^{2\pi} (\sin^2\theta)(-\sin\theta)d\theta - \cos^2\theta d\theta + 0d\theta \\ &= \int_0^{2\pi} (-\sin^3\theta - \cos^2\theta)d\theta \\ &= \int_0^{2\pi} (-\sin^3\theta)d\theta - \int_0^{2\pi} \cos^2\theta d\theta \\ &= -\frac{1}{2}\int_0^{2\pi}(1+\cos(2\theta))d\theta \\ &= -\frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Bigg|_0^{2\pi} \\ &= -\frac{1}{2}(2\pi - 0) \\ &= -\pi\end{aligned}$$

Question 6

Use Stokes' Theorem to calculate $\iint(\nabla\times\boldsymbol{F})\cdot\hat{n}dS$ for $\boldsymbol{F} = \langle xz^2, x^3, \cos(xz)\rangle$ where S is the part of the ellipsoid $xx^2 + y^2 + 3z^2 = 1$ below the xy-plane and $\hat{n}$ is the lower normal.

Solution

$$\begin{aligned}\iint_S (\nabla\times\vec{F})\cdot\hat{n}dS &= \oint_C \vec{F}\cdot\vec{T}dS \\ &= \int_C Mdx + Ndy + Pdz \\ &= \int_C xz^2dx + x^3dy + \cos(xz)dz\end{aligned}$$

Using polar coordinates,

$$\begin{aligned}z &= 0, & x &= \cos\theta, & y &= \sin\theta \\ dz &= 0, & dx &= -\sin\theta d\theta, & dy &= \cos\theta d\theta\end{aligned}$$

Hence,

$$\begin{aligned}&= \int_0^{2\pi} \cos\theta(0)(-\sin\theta d\theta) + \cos^3\theta(\cos\theta)d\theta + 0 \\ &= \int_0^{2\pi} \cos^4\theta d\theta = \frac{3\pi}{4}\end{aligned}$$

Question 7

Use Stoke's Theorem to evaluate the line integral $\int \boldsymbol{F} \cdot d\boldsymbol{r}c$ where $\boldsymbol{F}$ is the vector $\boldsymbol{F} = \left(4e^{x^2} - y\right)\boldsymbol{i} + \left(16\sin\left(y^2\right) + 3x\right)\boldsymbol{j} + \left(4y - 2x - e^z\right)\boldsymbol{k}$ and C is the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $z = 2x + 4y$ and C is oriented in a counterclockwise direction when viewed from above.

Solution

The curl of $\mathbf{F}$ is computed as,

$$\begin{aligned}\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4e^{x^2} - 1y & 16\sin(y^2) + 3x & 4y - 2x - e^z \end{vmatrix} \\ &= -4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}\end{aligned}$$

Now, writing the plane $z = 2x + 4y$ as the level surface $G(x, y, z) = -2x - 4y + z = 0$,

$$\mathbf{N}dS = \nabla G dA = \langle -2, -4, 1\rangle dA$$

Applying Stokes' theorem,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_S \int \langle -4, -2, 4\rangle \cdot \langle -2, -4, 1\rangle dA \\ &= \int_R \int [(-4)(-2) + (-2)(-4) + (4)(1)]dA \\ &= \int_R \int 20 dA \\ &= (20)(\text{Area of } R) \\ &= (20)(16\pi) = 320\pi\end{aligned}$$

Question 8

Evaluate the line integral of $\boldsymbol{F}(x, y, z) = \langle xy, 2z, 3y\rangle$ over the curve C that is the intersection of the cylinder $x^2 + y^2 = 9$ with the plane $x + z = 5$.

Solution

To describe the surface S enclosed by C, we use the parameterisation

$$x = u\cos v, \quad y = u\sin v, \quad z = 5 - u\cos v, \quad 0 \le u \le 3, \quad 0 \le v \le 2\pi$$

Using $\mathbf{r}_u = \langle \cos v, \sin v, -\cos v\rangle$ and $\mathbf{r}_v = \langle -u\sin v, u\cos v, u\sin v\rangle$, we obtain,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle u, 0, u\rangle$$

Compute the curl,

$$\text{curl } \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle xy, 2z, 3y\rangle = \langle 1, 0, x\rangle$$

Let D be the domain of the parameter,

$$D = \{(u, v) | 0 \le u \le 3, \quad 0 \le v \le 2\pi\}$$

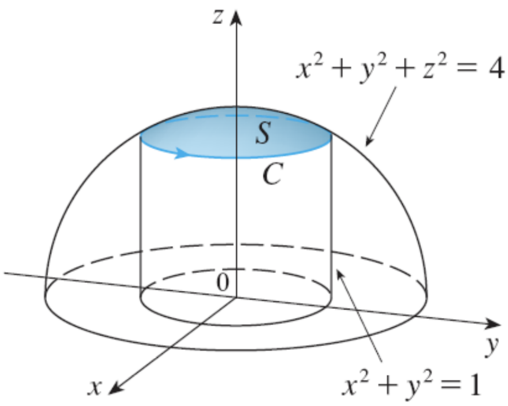
Using Green's Theorem,

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_D \text{curl } \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA \\
&= \int_0^3 \int_0^{2\pi} \langle 1, 0, u\cos v\rangle \cdot \langle u, 0, u\rangle dA \\
&= \int_0^3 \int_0^{2\pi} u + u^2 \cos v dv du \\
&= \int_0^3 (uv + u^2 \sin v)|_0^{2\pi} dv du \\
&= 2\pi \int_0^3 u dv du \\
&= 2\pi \left.\frac{u^2}{2}\right|_0^3 \\
&= 9\pi
\end{aligned}
$$

Question 9

Evaluate $\iint (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} dS$ where $\boldsymbol{F}(x,y,z) = \langle yz, xz, xy\rangle$ and S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.

Solution



Surface S is bounded by a circle formed by the intersection of the sphere of radius 2 and the cylinder of radius 1.

We can describe ∂S using the vector-valued function,

$$\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3}\rangle, \quad 0 \le t \le 2\pi$$

Using Stokes' Theorem,

$$
\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t, \sqrt{3}) \cdot \langle -\sin t, \cos t, 0\rangle dt \\
&= \sqrt{3} \int_0^{2\pi} (\cos^2 t - \sin^2 t) dt \\
&= \sqrt{3} \int_0^{2\pi} \cos 2t dt \\
&= 0
\end{aligned}
$$