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Tutorial 2: Partial Derivatives Engineering Applications of Partial Derivatives

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Question 1

Find the partial derivative $\left(\dfrac{\partial f}{\partial y}\right)$ and $\left(\dfrac{\partial f}{\partial x}\right)$ of these functions using the limit definition

(a) $f(x,y) = x^2y + 2x + y^3$

(b) $f(x,y) = x^2 - 4xy + y^2$

(c) $f(x,y) = 2x^3 + 3xy - y^2$

Solution

(a)

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2y + 2(x+h) + y^3 - \left(x^2y + 2x + y^3\right)}{h} \\
&= \lim_{h\to 0} \frac{x^2y + 2xhy + h^2y + 2x + 2h + y^3 - \left(x^2y + 2x + y^3\right)}{h} \\
&= \lim_{h\to 0} \frac{2xhy + h^2y + 2h}{h} \\
&= \lim_{h\to 0} 2xy + hy + 2 \\
&= 2xy + 2 \\
f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{x^2(y+h) + 2x + (y+h)^3 - x^2y - 2x - y^3}{h} \\
&= \lim_{h\to 0} \frac{x^2y + x^2h + 2x + y^3 + 3y^2h + 3yh^2 + h^3 - x^2y - 2x - y^3}{h} \\
&= \lim_{h\to 0} \frac{x^2h + 3y^2h + 3yh^2 + h^3}{h} \\
&= \lim_{h\to 0} x^2 + 3y^2 + 3yh + h^2 \\
&= x^2 + 3y^2
\end{aligned}$$

(b)

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - 4(x+h)y + y^2 - \left(x^2 - 4xy + y^2\right)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4xy - 4hy + y^2 - \left(x^2 - 4xy + y^2\right)}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 4hy}{h} \\
&= \lim_{h\to 0} (2x + h - 4y) \\
&= 2x - 4y
\end{aligned}$$

$$
\begin{aligned}
f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{x^2-4x(y+h)+(y+h)^2-\left(x^2-4xy+y^2\right)}{h} \\
&= \lim_{h\to 0} \frac{x^2-4xy-4xh+y^2+2yh+h^2-\left(x^2-4xy+y^2\right)}{h} \\
&= \lim_{h\to 0} \frac{-4xh+2yh+h^2}{h} \\
&= \lim_{h\to 0}(-4x+2y+h) \\
&= -4x+2y
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{h\left(6x^2+6xh+2h^2+3y\right)}{h} \\
&= \lim_{h\to 0} \left(6x^2+6xh+2h^2+3y\right) \\
&= 6x^2+3y \\
\frac{\partial f}{\partial y} &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{h(3x-2y-h)}{h} \\
&= \lim_{h\to 0}(3x-2y-h) \\
&= 3x-2y
\end{aligned}
$$

Question 2

Determine all the first and second order partial derivatives of the function

(a) $f(x,y) = x^2y^3+3y+x$

(b) $f(x,y) = x^4\sin 3y$

(c) $f(x,y) = x^2y+\ln\left(y^2-x\right)$

(d) $f(x,y) = e^{xy}(2x-y)$

Solution

(a)

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 2xy^3+1 & \frac{\partial f}{\partial y} &= 3x^2y^2+3 \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= 2y^3 & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= 6xy^2 \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= 6xy^2 & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= 6x^2y
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 4x^3\sin 3y & \frac{\partial f}{\partial y} &= 3x^4\cos 3y \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= 12x^2\sin 3y & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= 12x^3\cos 3y \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= 12x^3\cos 3y & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= -9x^4\sin 3y
\end{aligned}
$$

(c)

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xy - \frac{1}{y^2-x} & \frac{\partial f}{\partial y} &= x^2 + \frac{2y}{y^2-x} \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= 2y - \frac{1}{(y^2-x)^2} & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= 2x + \frac{2y}{(y^2-x)^2} \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= 2x + \frac{2y}{(y^2-x)^2} & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{-2y^2-2x}{(y^2-x)^2} \end{aligned}$$

(d)

$$\begin{aligned} \frac{\partial f}{\partial x} &= e^{xy}(2xy - y^2 + 2) & \frac{\partial f}{\partial y} &= e^{xy}(2x^2 - xy - 1) \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= e^{xy}(2xy^2 - y^3 + 4y) & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= e^{xy}(2x^3y - xy^2 + 4x - 2y) \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= e^{xy}(2x^2y - xy^2 + 4x - 2y) & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= e^{xy}(2x^3 - x^2y - 2x) \end{aligned}$$

Question 3

Find both partial derivatives for each of the following two variables functions

(a) $g(x,y) = ye^{x+y}$

(b) $h(x,y) = x\sin y - y\cos x$

(c) $p(x,y) = x^y + y^2$

(d) $U(x,y) = \frac{9y^3}{x-y}$

Solution

(a)

$$\begin{aligned} g(x,y) =& ye^{x+y} \\ g_x(x,y) &= ye^{x+y} \\ g_y(x,y) &= e^{x+y} + ye^{x+y} \end{aligned}$$

(b)

$$\begin{aligned} h(x,y) &= x\sin y - y\cos x \\ h_x(x,y) &= \sin y + y\sin x \\ h_y(x,y) &= x\cos y - \cos x \end{aligned}$$

(c)

$$\begin{aligned} p(x,y) =& x^y + y^2 \\ p_x(x,y) &= yx^{y-1} \\ p_y(x,y) &= x^y\ln x + 2y \end{aligned}$$

(d)

$$\begin{aligned} \frac{\partial U}{\partial x} = U_x = \frac{(x-y)(0) - 9y^3(1)}{(x-y)^2} &= \frac{-9y^3}{(x-y)^2} \\ \frac{\partial U}{\partial y} = U_y = \frac{(x-y)\left(27y^2\right) - 9y^3(-1)}{(x-y)^2} &= \frac{27xy^2 - 18y^3}{(x-y)^2} \end{aligned}$$

Question 4

For $f(x, y, z)$, use the implicit function theorem to find dy/dx and dy/dz

(a) $f(x, y, z) = x^2y^3 + z^2 + xyz$

(b) $f(x, y, z) = x^3z^2 + y^3 + 4xyz$

(c) $f(x, y, z) = 3x^2y^3 + xz^2y^2 + y^3zx^4 + y^2z$

Solution

(a)

$$\begin{aligned} f(x, y, z) &= x^2y^3 + z^2 + xyz \\ \frac{dy}{dx} &= -\frac{fx}{fy} = -\frac{2xy^3 + yz}{3x^2y^2 + xz} \\ \frac{dy}{dz} &= -\frac{fz}{fy} = -\frac{2z + xy}{3x^2y^2 + xz} \end{aligned}$$

(b)

$$\begin{aligned} f(x, y, z) &= x^3z^2 + y^3 + 4xyz \\ \frac{dy}{dx} &= -\frac{fx}{fy} = -\frac{3x^2z^2 + 4yz}{3y^2 + 4xz} \\ \frac{dy}{dz} &= -\frac{fz}{fy} = -\frac{2x^3z + 4xy}{3y^2 + 4xz} \end{aligned}$$

(c)

$$\begin{aligned} f(x, y, z) &= 3x^2y^3 + xz^2y^2 + y^3zx^4 + y^2z \\ \frac{dy}{dx} &= -\frac{fx}{fy} = -\frac{6xy^3 + z^2y^2 + 4y^3zx^3}{9x^2y^2 + 2xz^2y + 3y^2zx^4 + 2yz} \\ \frac{dy}{dz} &= -\frac{fz}{fy} = -\frac{2xzy^2 + y^3x^4 + y^2}{9x^2y^2 + 2xz^2y + 3y^2zx^4 + 2yz} \end{aligned}$$

Question 5

Find $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$, if applicable, for the following composite functions

(a) $F = \sin(x + y)$ where $x = 2st$ and $y = s^2 + t^2$

(b) $F = \ln\left(x^2 + y\right)$ where $x = \mathrm{e}^{(s+t2)}$ and $y = s^2 + t$

(c) $F = x^2y^2$ where $x = s\cos t$ and $y = s\sin t$

(d) $F = xy + y\mathrm{z}^2$ where $x = \mathrm{e}^t, y = \mathrm{e}^t \sin t$ and $\mathrm{z} = \mathrm{e}^t \cos t$

Solution

(a)

$$\begin{aligned} \frac{\partial F}{\partial s} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} \\ &= \cos(x + y)(2t) + \cos(x + y)(2s) \\ &= 2t\cos(2st + s^2 + t^2) + 2s\cos(st + s^2 + t^2) \\ &= 2\cos((s + t)^2)(s + t) \\ \frac{\partial F}{\partial t} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} \\ &= \cos(x + y)(2s) + \cos(x + y)(2t) \\ &= 2\cos((s + t)^2)(s + t) \end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial F}{\partial s} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} \\
&= \frac{2x}{x^2+y}e^{s+t^2} + \frac{1}{x^2+y}2s \\
&= \frac{2}{x^2+y}\left(xe^{s+t^2}+s\right) \\
\frac{\partial F}{\partial t} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} \\
&= \frac{2x}{x^2+y}2te^{s+t^2} + \frac{1}{x^2+y}(1) \\
&= \frac{1}{x^2+y}\left(4xte^{s+t^2}+1\right)
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\partial F}{\partial s} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} \\
&= 2xy^2\cos t + 2x^2y\sin t \\
&= 2(s\cos t)\left(s^2\sin^2 t\right)(\cos t) + 2\left(s^2\cos^2 t\right)(s\sin t)(\sin t) \\
&= 2s^3\cos^2 t\sin^2 t + 2s^3\cos^2 t\sin^2 t \\
&= 4s^3\cos^2 t\sin^2 t \\
&= 4s^3(\cos t\sin t)^2 \\
&= 4s^3\left(\frac{1}{2}\sin 2t\right)^2 \\
&= s^3(\sin 2t)^2 \\
\frac{\partial F}{\partial t} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} \\
&= 2xy^2(-s\sin t) + 2x^2y(s\cos t) \\
&= 2x^2ys\cos t - 2xy^2s\sin t \\
&= 2\left(s^2\cos^2 t\right)(s\sin t)s\cos t - 2(s\cos t)\left(s^2\sin^2 t\right)s\sin t \\
&= 2s^4\cos^3 t\sin t - 2s^4\cos t\sin^3 t \\
&= 2s^4(\cos t\sin t)\left(\cos^2 t - \sin^2 t\right) \\
&= 2s^4\left(\frac{1}{2}\sin 2t\right)(\cos 2t) \\
&= s^4\sin 2t\cos 2t
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{dF}{dt} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t} \\
&= ye^t + \left(x+z^2\right)\left(e^t\cos t + \sin^t e^t\right) + 2yz\left(-e^t\sin t + \cos te^t\right) \\
&= ye^t + xe^t\cos t + x\sin^t e^t + z^2e^t\cos t + z^2\sin te^t - 2yze^t\sin t + 2yz\cos te^t \\
&= ye^t + e^t\cos t\left(x+z^2+2yz\right) + e^t\sin t\left(x+z^2-2yz\right) \\
&= e^t\sin t\cdot e^t + e^t\cos t\left(e^t + e^{2t}\cos t + 2e^{2t}\sin t\cos t\right) \\
&\quad + e^t\sin t\left(e^t + e^{2t}\cos^2 t - 2e^{2t}\sin t\cos t\right) \\
&= e^{2t}\sin t + e^{2t}\cos t + e^{3t}\cos^3 t + 2e^{2t}\sin t\cos^2 t + e^{2t}\sin t + e^{3t}\sin t\cos^2 t \\
&\quad - 2e^3t\sin^2 t\cos t \\
&= 2e^{2t}\sin t + e^{2t}\cos t + e^{3t}\cos^3 t + 3e^{3t}\sin t\cos^2 t - 2e^{3t}\sin^2 t\cos t \\
&= e^{2t}(2\sin t + \cos t) + e^{3t}\left(\cos^3 t + 3\sin t\cos^2 t - 2\sin^2 t\cos t\right)
\end{aligned}$$

Question 6

Find dy/dx and dy/dz (if applicable) for each of the following

(a) $7x^2 + 2xy^2 + 9y^4 = 0$

(b) $x^3z^2 + y^3 + 4xyz = 0$

(c) $3x^2y^3 + xz^2y^2 + y^3zx^4 + y^2z = 0$

(d) $y^5 + x^2y^3 = 1 + y\exp\left(x^2\right)$

Solution

(a)

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{14x + 2y^2}{36y^2 + 4xy}$$

(b)

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2z^2 + 4yz}{3y^2 + 4xz}$$
$$\frac{dy}{dz} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}} = -\frac{2x^3z + 4xy}{3y^2 + 4xz}$$

(c)

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{6xy^3 + z^2y^2 + 4y^3zx^3}{9x^2y^2 + 2xz^2y + 3y^2zx^4 + 2yz}$$
$$\frac{dy}{dz} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}} = -\frac{2xzy^2 + y^3x^4 + y^2}{9x^2y^2 + 2xz^2y + 3y^zx^4 + 2yz}$$

(d)

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2xy\left(y^2 - e^{x^2}\right)}{5y^4 + 3x^2y^2 - e^{x^2}}$$

Question 7

(a) In polar coordinates, $x = r\cos\theta, y = r\sin\theta$, show that $\frac{\partial(x,y)}{\partial(r,\theta)} = r$

(b) Obtain the Jacobian J of the transformation $s = 2x + y, t = x - 2y$ and determine the inverse of the transformation J_1. Confirm that $J_1 = J^{-1}$.

(c) Show that if $x + y = \mathrm{u}$ and $y = \mathrm{uv}$, then $\frac{\partial(x,y)}{\partial(u,v)} = u$.

(d) Verify whether the functions $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$ are functionally dependent.

(e) If $x = \mathrm{uv}, \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution

(a)

$$\text{For } x = r\cos\theta, \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$y = r\sin\theta, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta$$
$$\therefore \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

(b)

$$\begin{aligned} J &= \frac{\partial(s,t)}{\partial(x,y)} \\ &= \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \\ &= -5 \end{aligned}$$

$$x = \tfrac{1}{5}(2s+t) \qquad y = \tfrac{1}{5}(s-2t)$$

$$\begin{aligned} J_1 &= \frac{\partial(x,y)}{\partial(s,t)} \\ &= \begin{vmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{vmatrix} \\ &= -\frac{1}{5} \end{aligned}$$

$$\therefore J_1 = J^{-1}$$

(c)

$$\begin{aligned} &x + uv = u \Longrightarrow x = u - uv \\ &\tfrac{dx}{du} = 1 - v \\ &\tfrac{dx}{dv} = -1 \end{aligned}$$

$$\begin{aligned} &y = uv \\ &\tfrac{dy}{du} = v \\ &\tfrac{dy}{dv} = u \end{aligned}$$

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} \\ &= u - uv + uv \\ &= u \end{aligned}$$

(d)

$$\begin{aligned} \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} \\ &= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} \\ &= 0 \end{aligned}$$

(e)

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -\frac{2v}{(u-v)^2} & -\frac{2u}{(u-v)^2} \end{vmatrix} \\ &= \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} \\ &= \frac{4uv}{(u-v)^2} \\ \frac{\partial(u,v)}{\partial(x,y)} &= \frac{(u-v)^2}{4uv} \end{aligned}$$

Question 8

(a) How sensitive is the volume $V = \pi r^2 h$ of a right circular cylinder to small changes in its radius and height near the point $(r_0, h_0) = (1, 3)$?

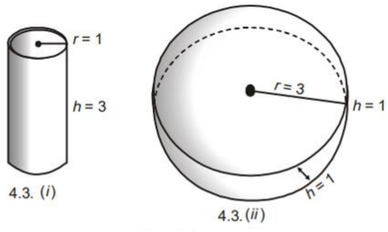
(b) If r is measured with an accuracy of $\pm 1\%$ and h with an accuracy of $\pm 0.5\%$, about how accurately can V be calculated from the formula $V = \pi r^2 h$?

(c) The period T of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$, find the maximum percentage error in T due to possible errors up to 1% in l and 2% in g (Hint: $\frac{dl}{l} = 0.001$ and $\frac{dg}{g} = 0.002$)

(d) The range R of a projectile which starts with a velocity v at an elevation α is given by $R = \frac{v^2 \sin 2\alpha}{g}$. Find the percentage error in R due to an error of 1% in v and an error of 0.5% in α.

Solution

(a)



By increment method, we get

$$\begin{aligned}
\Delta V &\approx \left(\frac{\partial V}{\partial r}\right)_{(r_0,h_0)} \Delta r + \left(\frac{\partial V}{\partial h}\right)_{(r_0,h_0)} \Delta h \\
&= V_r(1,3)\Delta r + V_h(1,3)\Delta h \\
&= (2\pi r h)_{(1,3)}\Delta r + \left(\pi r^2\right)_{(1,3)} \cdot \Delta h \\
&= 6\pi \cdot \Delta r + \pi \cdot \Delta h
\end{aligned}$$

The above result shows that a one-unit change in r will change V nearly by 6π units and a one-unit change in h will change V nearly by π units.

Therefore, the volume of a cylinder with radius $r = 1$ and height $h = 3$ is nearly 6 times as sensitive to small change in r as it is to small change in h Fig. 4.3(i).

In contrast, if value of r and h are reversed, so that $r = 3$ and $h = 1$, then $\Delta V \approx 6\pi \cdot \Delta r + 9\pi.\ \Delta h$. The volume is now more sensitive to small change in h to that it is to change in r. Thus the sensitivity to change depends not only on the increment but also on the relative size of r and h (See Fig. 4.3 (ii)).

(b) For $V = \pi r^2 h$ or $\log V = \log \pi + 2\log r + \log h$, increment approximation implies

$$\frac{\Delta V}{V} \approx 2\frac{\Delta r}{r} + \frac{\Delta h}{h}.$$

Given, $\left.\begin{array}{l} \frac{\Delta r}{r} \times 100 = \pm 1, \\ \frac{\Delta h}{h} \times 100 = \pm 0.5 = \frac{1}{2} \end{array}\right\}$ so that $\left.\begin{array}{l} \left|\frac{\Delta r}{r}\right| \le \frac{1}{100} \\ \left|\frac{\Delta h}{h}\right| \le \frac{1}{200} \end{array}\right\}$

$$\therefore \left|\frac{\Delta V}{V}\right| \approx \left|2\frac{\Delta r}{r} + \frac{\Delta h}{h}\right| \le 2\left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta h}{h}\right| = \frac{2}{100} + \frac{1}{200} = 0.025$$

Thus, the maximum percentage error (or change), viz. $\left|\frac{\Delta V}{V}\right| \times 100$, due to possible percentage error (or change) in measurement of r and A will be about 2.5 per cent.

(c)

$$\begin{aligned}
\log T &= \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g \\
\frac{dT}{T} &= 0 + \frac{1}{2l}dl - \frac{1}{2g}dg \\
&= \left(\frac{1}{2}\right)\left(\frac{dl}{l}\right) - \left(\frac{1}{2}\right)\left(\frac{dg}{g}\right) \\
&= \left(\frac{1}{2}\right)(0.001) \pm \left(\frac{1}{2}\right)(0.002) \\
&= 0.0005 \pm 0.001 \\
&\therefore \text{Max error} = \ 1.5\%
\end{aligned}$$

(d) Given $R = \frac{v^2 \sin 2\alpha}{g}$ or $\log R = 2\log v + \log\sin 2\alpha - \log g$.
By error approximation (i.e. taking differential on both sides)

$$\frac{\delta R}{R} = 2\frac{\delta v}{v} + \frac{1}{\sin 2\alpha}\cos 2\alpha \cdot 2\delta\alpha, (\delta g = 0)$$

or

$$\begin{aligned}\left(\frac{\delta R}{R}\times 100\right) &= 2\left(\frac{\delta v}{v}\times 100\right) + 2\alpha(\cot 2\alpha)\left(\frac{\delta\alpha}{\alpha}\times 100\right)\\ &= 2\times 1 + 2\alpha\cdot\cot 2\alpha\times 0.5 = 2 + 2\alpha\cot 2\alpha\times\frac{1}{2}\\ &= (2+\alpha\cot 2\alpha)\end{aligned}$$

Hence, the percentage error in calculation of R due to errors of 1% in R and 0.5% in α is $(2+\alpha\cot 2\alpha)$.

Question 9

Find the equations of the tangent plane and normal line to the following surfaces at the points indicated:

(a) $x^2 + 2y^2 + 3z^2 = 6$ at $(1, 1, 1)$

(b) $2x^2 + y^2 - z^2 = -3$ at $(1, 2, 3)$

(c) $x^2 + y^2 - z = 1$ at $(1, 2, 4)$

(d) $\ln\left(\frac{x}{y}\right) - z^2(x-2y) - 3z = 3$ at $(4, 2, -1)$

(e) $x^3z + z^3x - 2yz = 0$ at $(1, 1, 1)$

(f) $z = 5 + (x-1)^2 + (y+2)^2$ at $(2, 0, 10)$

(g) $\frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4} = 1$ at $(1, 2, 1)$

(h) $ze^x + e^{z+1} + xy + y = 3$ at $(0, 3-1)$

Solution

(a) Partial derivation yields:

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 4y \qquad \frac{\partial f}{\partial z} = 6z$$

$$\implies \frac{\partial f}{\partial x}_{(1,1,1)} = 2 \qquad \frac{\partial f}{\partial y}_{(1,1,1)} = 4 \qquad \frac{\partial f}{\partial z}_{(1,1,1)} = 6$$

Tangent plane:

$$\begin{aligned}(x-x_o)\left(\frac{\partial f}{\partial x}\right)_0 + (y-y_0)\left(\frac{\partial f}{\partial y}\right)_0 + (z-z_0)\left(\frac{\partial f}{\partial z}\right)_0 &= 0\\ 2(x-1)+4(y-1)+6(z-1) &= 0\\ 2x-2+4y-4+6z-6 &= 0\\ x+2y+3z &= 6\end{aligned}$$

Normal line:

$$\begin{aligned}\frac{x-x_0}{\left(\frac{\partial f}{\partial x}\right)_0} &= \frac{y-y_0}{\left(\frac{\partial f}{\partial y}\right)_0} = \frac{z-z_0}{\left(\frac{\partial f}{\partial z}\right)_0}\\ \frac{x-1}{2} &= \frac{y-1}{4} = \frac{z-1}{6}\\ \frac{x-1}{1} &= \frac{y-1}{2} = \frac{z-1}{3}\end{aligned}$$

(b) Partial derivation yields:

$$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = 2y \qquad \frac{\partial f}{\partial z} = -2z$$

$$\Longrightarrow \left.\frac{\partial f}{\partial x}\right._{(1,2,3)} = 4 \qquad \left.\frac{\partial f}{\partial y}\right._{(1,2,3)} = 4 \qquad \left.\frac{\partial f}{\partial z}\right._{(1,2,3)} = -6$$

Tangent plane:

$$\begin{aligned}(x - x_o)\left(\frac{\partial f}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_0 + (z - z_0)\left(\frac{\partial f}{\partial z}\right)_0 &= 0\\ 4(x-1) + 4(y-2) - 6(z-3) &= 0\\ 4x - 4 + 4y - 8 - 6z + 18 &= 0\\ 4x + 4y - 6z &= -6\\ 2x + 2y - 3z &= -3\end{aligned}$$

Normal line:

$$\begin{aligned}\frac{x - x_0}{\left(\frac{\partial f}{\partial x}\right)_0} &= \frac{y - y_0}{\left(\frac{\partial f}{\partial y}\right)_0} = \frac{z - z_0}{\left(\frac{\partial f}{\partial z}\right)_0}\\ \frac{x-1}{4} &= \frac{y-2}{4} = \frac{z-3}{-6}\\ \frac{x-1}{2} &= \frac{y-2}{2} = \frac{z-3}{-3}\end{aligned}$$

(c) Partial derivation yields:

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y \qquad \frac{\partial f}{\partial z} = -1$$

$$\Longrightarrow \left.\frac{\partial f}{\partial x}\right._{(1,2,4)} = 2 \qquad \left.\frac{\partial f}{\partial y}\right._{(1,2,4)} = 4 \qquad \left.\frac{\partial f}{\partial z}\right._{(1,2,4)} = -1$$

Tangent plane:

$$\begin{aligned}(x - x_o)\left(\frac{\partial f}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_0 + (z - z_0)\left(\frac{\partial f}{\partial z}\right)_0 &= 0\\ 2(x-1) + 4(y-2) - 1(z-4) &= 0\\ 2x - 2 + 4y - 8 - z + 4 &= 0\\ 2x + 4y - z &= 6\end{aligned}$$

Normal line:

$$\begin{aligned}\frac{x - x_0}{\left(\frac{\partial f}{\partial x}\right)_0} &= \frac{y - y_0}{\left(\frac{\partial f}{\partial y}\right)_0} = \frac{z - z_0}{\left(\frac{\partial f}{\partial z}\right)_0}\\ \frac{x-1}{2} &= \frac{y-2}{4} = \frac{z-4}{-1}\end{aligned}$$

(d) Partial derivation yields:

$$\frac{\partial f}{\partial x} = \frac{1}{x} - z^2 \qquad \frac{\partial f}{\partial y} = 2z^2 - \frac{1}{y} \qquad \frac{\partial f}{\partial z} = -2z(x - 2y) - 3$$

$$\Longrightarrow \left.\frac{\partial f}{\partial x}\right._{(4,2,-1)} = -\frac{3}{4} \qquad \left.\frac{\partial f}{\partial y}\right._{(4,2,-1)} = \frac{3}{2} \qquad \left.\frac{\partial f}{\partial z}\right._{(4,2,-1)} = -3$$

Tangent plane:

$$
\begin{aligned}
(x - x_o)\left(\frac{\partial f}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_0 + (z - z_0)\left(\frac{\partial f}{\partial z}\right)_0 &= 0 \\
-\frac{3}{4}(x-4) + \frac{3}{2}(y-2) - 3(z+1) &= 0 \\
-\frac{3}{4}x + 3 + \frac{3}{2}y - 3 - 3z - 3 &= 0 \\
-\frac{3}{4}x + \frac{3}{2}y - 3z &= 3
\end{aligned}
$$

Normal line:

$$
\begin{aligned}
\frac{x - x_0}{\left(\frac{\partial f}{\partial x}\right)_0} = \frac{y - y_0}{\left(\frac{\partial f}{\partial y}\right)_0} = \frac{z - z_0}{\left(\frac{\partial f}{\partial z}\right)_0} \\
\frac{x-4}{-\frac{3}{4}} = \frac{y-2}{\frac{3}{2}} = \frac{z+1}{-3}
\end{aligned}
$$

(e) Partial derivation yields:

$$
\frac{\partial f}{\partial x} = 3x^2 z + z^3 \qquad \frac{\partial f}{\partial y} = -2z \qquad \frac{\partial f}{\partial z} = x^3 + 3z^2 x - 2y
$$

$$
\implies \frac{\partial f}{\partial x}_{(1,1,1)} = 4 \qquad \frac{\partial f}{\partial y}_{(1,1,1)} = -2 \qquad \frac{\partial f}{\partial z}_{(1,1,1)} = 2
$$

Tangent plane:

$$
\begin{aligned}
(x - x_o)\left(\frac{\partial f}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_0 + (z - z_0)\left(\frac{\partial f}{\partial z}\right)_0 &= 0 \\
4(x-1) - 2(y-1) + 2(z-1) &= 0 \\
4x - 4 - 2y + 2 + 2z - 2 &= 0 \\
4x - 2y + 2z &= 4 \\
2x - y + z &= 2
\end{aligned}
$$

Normal line:

$$
\begin{aligned}
\frac{x - x_0}{\left(\frac{\partial f}{\partial x}\right)_0} = \frac{y - y_0}{\left(\frac{\partial f}{\partial y}\right)_0} = \frac{z - z_0}{\left(\frac{\partial f}{\partial z}\right)_0} \\
\frac{x-1}{4} = \frac{y-1}{-2} = \frac{z-1}{2} \\
\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z-1}{1}
\end{aligned}
$$

(f)

$$
z = 5 + (x-1)^2 + (y+2)^2 \implies (x-1)^2 + (y+2)^2 - z = 5
$$

Partial derivation yields:

$$
\frac{\partial f}{\partial x} = 2(x-1) \qquad \frac{\partial f}{\partial y} = 2(y+2) \qquad \frac{\partial f}{\partial z} = -1
$$

$$
\implies \frac{\partial f}{\partial x}_{(2,0,10)} = 2 \qquad \frac{\partial f}{\partial y}_{(2,0,10)} = 4 \qquad \frac{\partial f}{\partial z}_{(2,0,10)} = -1
$$

Tangent plane:

$$\begin{aligned}(x-x_o)\left(\frac{\partial f}{\partial x}\right)_0+(y-y_0)\left(\frac{\partial f}{\partial y}\right)_0+(z-z_0)\left(\frac{\partial f}{\partial z}\right)_0&=0\\ 2(x-2)+4(y)-(z-10)&=0\\ 2x-4+4y-z+10&=0\\ 2x+4y-z&=-6\end{aligned}$$

Normal line:

$$\begin{aligned}\frac{x-x_0}{\left(\frac{\partial f}{\partial x}\right)_0}&=\frac{y-y_0}{\left(\frac{\partial f}{\partial y}\right)_0}=\frac{z-z_0}{\left(\frac{\partial f}{\partial z}\right)_0}\\ \frac{x-2}{2}&=\frac{y}{4}=\frac{z-10}{-1}\end{aligned}$$

(g) Partial derivation yields:

$$\frac{\partial f}{\partial x}=\frac{x}{6}\qquad\frac{\partial f}{\partial y}=\frac{y}{3}\qquad\frac{\partial f}{\partial z}=\frac{z}{2}$$

$$\implies\frac{\partial f}{\partial x}_{(1,2,1)}=\frac{1}{6}\qquad\frac{\partial f}{\partial y}_{(1,2,1)}=\frac{2}{3}\qquad\frac{\partial f}{\partial z}_{(1,2,1)}=\frac{1}{2}$$

Tangent plane:

$$\begin{aligned}(x-x_o)\left(\frac{\partial f}{\partial x}\right)_0+(y-y_0)\left(\frac{\partial f}{\partial y}\right)_0+(z-z_0)\left(\frac{\partial f}{\partial z}\right)_0&=0\\ \frac{1}{6}(x-1)+\frac{2}{3}(y-2)+\frac{1}{2}(z-1)&=0\\ \frac{1}{6}x-\frac{1}{6}+\frac{2}{3}y-\frac{4}{3}+\frac{1}{2}z-\frac{1}{2}&=0\\ \frac{1}{6}x+\frac{4}{6}y+\frac{3}{6}z-\frac{1}{6}-\frac{8}{6}-\frac{3}{6}&=0\\ \frac{1}{6}x+\frac{4}{6}y+\frac{3}{6}z-2&=0\\ x+4y+3z&=12\end{aligned}$$

Normal line:

$$\begin{aligned}\frac{x-x_0}{\left(\frac{\partial f}{\partial x}\right)_0}&=\frac{y-y_0}{\left(\frac{\partial f}{\partial y}\right)_0}=\frac{z-z_0}{\left(\frac{\partial f}{\partial z}\right)_0}\\ \frac{x-1}{\frac{1}{6}}&=\frac{y-2}{\frac{2}{3}}=\frac{z-1}{\frac{1}{2}}\\ \frac{x-1}{\frac{1}{6}}&=\frac{y-2}{\frac{4}{6}}=\frac{z-1}{\frac{3}{6}}\\ \frac{x-1}{1}&=\frac{y-2}{4}=\frac{z-1}{3}\end{aligned}$$

(h) Partial derivation yields:

$$\frac{\partial f}{\partial x}=ze^x+y\qquad\frac{\partial f}{\partial y}=x+1\qquad\frac{\partial f}{\partial z}=e^x+e^{z+1}$$

$$\implies\frac{\partial f}{\partial x}_{(0,3,-1)}=2\qquad\frac{\partial f}{\partial y}_{(0,3,-1)}=1\qquad\frac{\partial f}{\partial z}_{(0,3,-1)}=2$$

Tangent plane:

$$\begin{aligned}(x-x_o)\left(\frac{\partial f}{\partial x}\right)_0+(y-y_0)\left(\frac{\partial f}{\partial y}\right)_0+(z-z_0)\left(\frac{\partial f}{\partial z}\right)_0&=0\\ 2(x)+(y-3)+2(z+1)&=0\\ 2x+y-3+2z+2&=0\\ 2x+y+2z&=1\end{aligned}$$

Normal line:

$$\begin{gathered}\frac{x-x_0}{\left(\frac{\partial f}{\partial x}\right)_0}=\frac{y-y_0}{\left(\frac{\partial f}{\partial y}\right)_0}=\frac{z-z_0}{\left(\frac{\partial f}{\partial z}\right)_0}\\ \frac{x}{2}=\frac{y-3}{1}=\frac{z+1}{2}\end{gathered}$$