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Tutorial 3: Vector Algebra I

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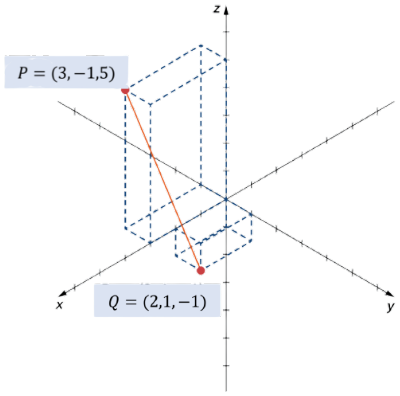
Question 1

Sketch the two points $P(3, -1, 5)$ and $Q(2, 1, -1)$ in three-dimensional space and find the distance between the two points.

Solution



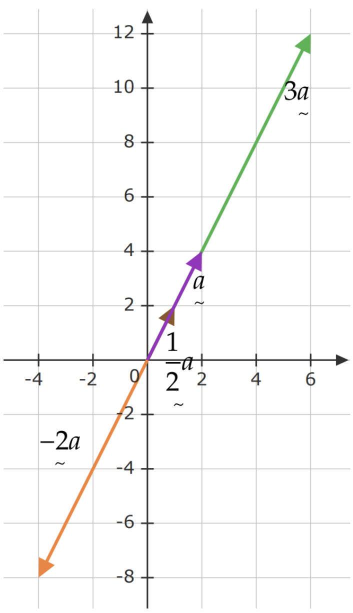
Distance between the two points:

$$
\begin{aligned}
|\overrightarrow{PQ}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\
&= \sqrt{41}
\end{aligned}
$$

Question 2

For the position vector $\underset{\sim}{a} = \langle 2, 4 \rangle$, compute $3\underset{\sim}{a}$, $\frac{1}{2}\underset{\sim}{a}$, and $-2\underset{\sim}{a}$. Sketch all four vectors on the same axis system. Discuss the effects of scalar multiplication on the magnitude and direction of the original vector.

Solution



- The scalar multiplication affects the magnitude of vectors if the scalar is a negative value. For example, $-2\underset{\sim}{a}$ switch the direction in exactly the opposite direction.
- The scalar multiplication affects the magnitude of vectors if the scalar is not equal to 1.
- When scalar > 1, the length of the vector is stretched. (magnitude increased)
- When scalar < 1, the length of the vector is shrinked. (magnitude decteased)

Question 3

Two vectors are given as $\overrightarrow{OP} = \underset{\sim}{i} + 3\underset{\sim}{j} - 7\underset{\sim}{k}$ and $\overrightarrow{OQ} = 5\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$

(i) Find the unit vector in the direction of $\overrightarrow{PQ}$

(ii) Find the direction cosines of $\overrightarrow{PQ}$

(iii) Find the vector with magnitude of 5 in the direction of $\overrightarrow{QP}$ in polar form

Solution

(i)

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\ &= \langle 5, -2, 4\rangle - \langle 1, 3, -7\rangle\\ &= \langle 4, -5, 11\rangle\end{aligned}$$

(ii)

$$\begin{aligned}\overrightarrow{PQ} &= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}\\ &= \frac{\langle 4, -5, 11\rangle}{\sqrt{(4)^2 + (-5)^2 + (11)^2}}\\ &= \frac{1}{9\sqrt{2}}\langle 4, -5, 11\rangle\end{aligned}$$

(iii) Comparing each component, we obtain:

- For component $\underset{\sim}{i}$, $\cos\alpha = \frac{4}{9\sqrt{2}}$
- For component $\underset{\sim}{j}$, $\cos\beta = \frac{-5}{9\sqrt{2}}$
- For component $\underset{\sim}{k}$, $\cos\gamma = \frac{11}{9\sqrt{2}}$

Question 4

Two points are given as $A(1, 2)$ and $B(3, 4)$.

(i) Find the vector equation of line L that is passing through point A and B.

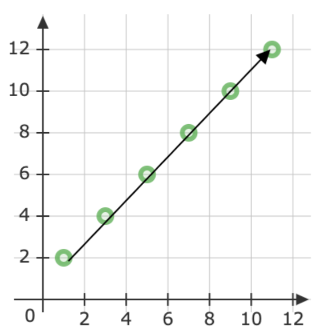
(ii) Sketch the line for $t = 0 : 1 : 5$ and indicate its direction and initial point.

Solution

(i) Direction of vector is parallel with $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 3, 4\rangle - \langle 1, 2\rangle = \langle 2, 2\rangle$
Vector equation of line L is $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v} = \langle 1, 2\rangle + t\langle 2, 2\rangle$

(ii)

t	$\underset{\sim}{r}$
0	$\langle 1, 2\rangle$
1	$\langle 3, 4\rangle$
2	$\langle 5, 6\rangle$
3	$\langle 7, 8\rangle$
4	$\langle 9, 10\rangle$
5	$\langle 11, 12\rangle$



Question 5

If a unit vector $\vec{a}$ makes an angle of $\pi/3$ with $i, \pi/4$ with j and acute angle θ with k, find θ and the components of $\vec{a}$.

Solution

Let α, β, γ to represent angles of $\vec{a}$, and $\vec{a} = a_1 i + a_2 j + a_3 k$.

$$\begin{aligned}
\cos\alpha &= \frac{a_1}{|a_1|} \\
\cos\left(\frac{\pi}{3}\right) &= \frac{a_1}{1} \\
a_1 &= \frac{1}{2} \\
\cos\beta &= \frac{a_2}{|a_2|} \\
\cos\left(\frac{\pi}{4}\right) &= \frac{a_1}{1} \\
a_1 &= \frac{1}{\sqrt{2}} \\
\cos\gamma &= \frac{a_3}{|a_3|} \\
\cos\theta &= \frac{a_3}{1} \\
a_3 &= \cos\theta
\end{aligned}$$

$$\therefore \ \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \cos\theta k$$

$$\begin{aligned}
|\vec{a}| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (\cos\theta)^2} \\
1 &= \sqrt{\frac{1}{4} + \frac{1}{2} + \cos^2\theta} \\
1 &= \frac{3}{4} + \cos^2\theta \\
\cos^2\theta &= \frac{1}{4} \\
\cos\theta &= \pm\frac{1}{2} \\
\theta &= 60^\circ \ or \ 120^\circ
\end{aligned}$$

$$\therefore \ \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \frac{1}{2}k \quad or \quad \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j - \frac{1}{2}k$$

Question 6

If $\vec{a}$ is a unit vector and $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.

Solution Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

$$\begin{aligned}
(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) &= 8 \\
\vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{a} &= 8 \\
|\vec{x}|^2 - |\vec{a}|^2 &= 8 \\
|\vec{x}|^2 - 1 &= 8 \\
|\vec{x}|^2 &= 9 \\
|\vec{x}| &= 3
\end{aligned}$$

Note that magnitude of vector is non-negative.

Question 7

Find the gradient for $f = \left(2x^2 + y\right) / \left(x^2 - y^2\right)$.

Solution

$$\frac{d}{dx}(f) = \frac{\left(x^2 - y^2\right)(4x) - \left(2x^2 + y\right)(2x)}{\left(x^2 - y^2\right)^2}$$

$$\frac{d}{dx}(f) = \frac{2xy(-2y - 1)}{\left(x^2 - y^2\right)^2}$$

and

$$\frac{d}{dy}(f) = \frac{\left(x^2 - y^2\right)(1) - \left(2x^2 + y\right)(-2y)}{\left(x^2 - y^2\right)^2}$$

$$\frac{d}{dy}(f) = \frac{x^2 + 4x^2y + y^2}{\left(x^2 - y^2\right)^2}$$

Hence

$$\nabla f = \frac{2xy(-2y - 1)}{\left(x^2 - y^2\right)^2}\boldsymbol{i} + \frac{x^2 + 4x^2y + y^2}{\left(x^2 - y^2\right)^2}\boldsymbol{j}$$

Question 8

Calculate the divergence of the following vector fields of $F(x, y)$ and $G(x, y)$;

(a) $F = y^3\boldsymbol{i} + xy\boldsymbol{j}$

(b) $G = \frac{4y}{x^2}\boldsymbol{i} + \sin(y)\boldsymbol{j} + 3\boldsymbol{k}$

(c) $G = e^x\boldsymbol{i} + \ln(xy)\boldsymbol{j} + e^{xyz}\boldsymbol{k}$

Solution

(a)

$$\begin{aligned}\nabla \cdot \boldsymbol{F}(x, y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}y^3 + \frac{\partial}{\partial y}xy = 0 + x = x\end{aligned}$$

(b)

$$\begin{aligned}\nabla \cdot \boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= \frac{\partial}{\partial x}\left(\frac{4y}{x^2}\right) + \frac{\partial}{\partial y}\sin(y) + \frac{\partial}{\partial z}3 \\ &= 4y \times \frac{\partial}{\partial x}x^{-2} + \cos(y) = 4y \times (-2)x^{-2-1} + \cos(y) \\ &= -8yx^{-3} + \cos(y) = -\frac{8y}{x^3} + \cos(y)\end{aligned}$$

(c)

$$\begin{aligned}\nabla \cdot \boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= \frac{\partial}{\partial x}\mathrm{e}^x + \frac{\partial}{\partial y}\ln(xy) + \frac{\partial}{\partial z}\mathrm{e}^{xyz} \\ &= \mathrm{e}^x + \frac{\partial}{\partial y}(\ln(x) + \ln(y)) + \mathrm{e}^{xyz} \times \frac{\partial}{\partial z}(xyz) \\ &= \mathrm{e}^x + \frac{1}{y} + xy\mathrm{e}^{xyz}\end{aligned}$$

Question 9

Calculate the curl of the following vector fields of $F(x, y, z)$;

(a) $F = 3x^2\boldsymbol{i} + 2z\boldsymbol{j} - x\boldsymbol{k}$

(b) $F = y^3\boldsymbol{i} + xy\boldsymbol{j} - z\boldsymbol{k}$

(c) $F = \left(1 + y + z^2\right)\boldsymbol{i} + \left(e^{xyz}\right)\boldsymbol{j} - (xyz)\boldsymbol{k}$

Solution

(a)

$$\begin{aligned}
\nabla \times \boldsymbol{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 2z & -x \end{vmatrix} \\
&= \left(\frac{\partial(-x)}{\partial y} - \frac{\partial(2z)}{\partial z}\right) i - \left(\frac{\partial(-x)}{\partial x} - \frac{\partial\left(3x^2\right)}{\partial z}\right) j + \left(\frac{\partial(2z)}{\partial x} - \frac{\partial\left(3x^2\right)}{\partial y}\right) k \\
&= (0 - 2)i - (-1 - 0)j + (0 - 0)k \\
&= -2i + j
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla \times \boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \boldsymbol{k} \\
&= \left(\frac{\partial(-z)}{\partial y} - \frac{\partial(xy)}{\partial z}\right) i - \left(\frac{\partial(-z)}{\partial x} - \frac{\partial\left(y^3\right)}{\partial z}\right) j + \left(\frac{\partial(xy)}{\partial x} - \frac{\partial\left(y^3\right)}{\partial y}\right) \boldsymbol{k} \\
&= 0i - 0j + \left(y - 3y^2\right) \boldsymbol{k} \\
&= \left(y - 3y^2\right) \boldsymbol{k}
\end{aligned}$$

i.e., the curl vector is in the k direction.

(c)

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = &\left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}\left(e^{xyz}\right)\right)\vec{i} + \left(\frac{\partial}{\partial z}\left(1 + y + z^2\right) - \frac{\partial}{\partial x}(xyz)\right)\vec{j} \\
&+ \left(\frac{\partial}{\partial x}\left(e^{xyz}\right) - \frac{\partial}{\partial y}\left(1 + y + z^2\right)\right)\vec{k} \\
\operatorname{curl}(\mathbf{F}) = &\left(xz - xye^{xyz}\right)\vec{i} + (2z - yz)\vec{k} + \left(yze^{xyz} - 1\right)\vec{k}
\end{aligned}$$