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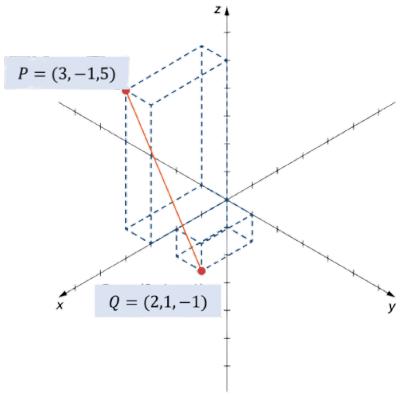
Tutorial 3: Vector Algebra I

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Question 1

Sketch the two points $P(3,-1,5)$ and $Q(2,1,-1)$ in three-dimensional space and find the distance between the two points.

Solution



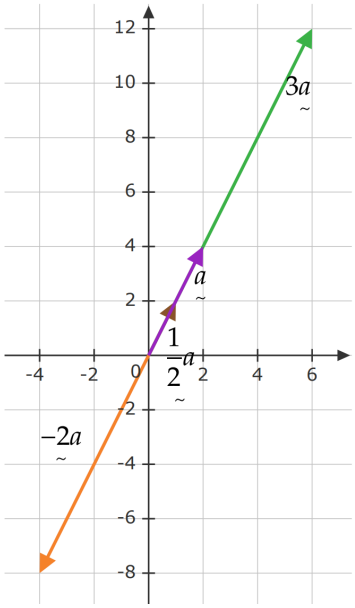
Distance between the two points:

$$\begin{aligned}|\overrightarrow{PQ}| &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \\ &= \sqrt{(2-3)^2+(1-(-1))^2+(-1-5)^2} \\ &= \sqrt{41}\end{aligned}$$

Question 2

For the position vector $\underset{\sim}{a} = \langle 2,4\rangle$, compute $3\underset{\sim}{a}$, $\frac{1}{2}\underset{\sim}{a}$, and $-2\underset{\sim}{a}$. Sketch all four vectors on the same axis system. Discuss the effects of scalar multiplication on the magnitude and direction of the original vector.

Solution



- Multiplication by a negative scalar reverses the direction of a nonzero vector. For example, $-2\underset{\sim}{a}$ points in the opposite direction and has twice the magnitude.
- Multiplication by a scalar c multiplies the magnitude by $|c|$. The magnitude is unchanged when $c = 1$ or $c = -1$.
- When $|c| > 1$, the vector is stretched.
- When $0 < |c| < 1$, the vector is shortened. When $c = 0$, the result is the zero vector, which has no direction.

Question 3

Two vectors are given as $\overrightarrow{OP} = \underset{\sim}{i} + 3\underset{\sim}{j} - 7\underset{\sim}{k}$ and $\overrightarrow{OQ} = 5\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$

(i) Find the unit vector in the direction of $\overrightarrow{PQ}$

(ii) Find the direction cosines of $\overrightarrow{PQ}$

(iii) Find the vector with magnitude of 5 in the direction of $\overrightarrow{QP}$ in polar form

Solution

(i)

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \langle 5, -2, 4\rangle - \langle 1, 3, -7\rangle \\ &= \langle 4, -5, 11\rangle\end{aligned}$$

(ii)

$$\begin{aligned}\hat{u}_{PQ} &= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} \\ &= \frac{\langle 4, -5, 11\rangle}{\sqrt{(4)^2 + (-5)^2 + (11)^2}} \\ &= \frac{1}{9\sqrt{2}}\langle 4, -5, 11\rangle\end{aligned}$$

(iii) Comparing each component, we obtain:

- For component $\underset{\sim}{i}$, $\cos\alpha = \frac{4}{9\sqrt{2}}$
- For component $\underset{\sim}{j}$, $\cos\beta = \frac{-5}{9\sqrt{2}}$
- For component $\underset{\sim}{k}$, $\cos\gamma = \frac{11}{9\sqrt{2}}$

Question 4

Two points are given as $A(1, 2)$ and $B(3, 4)$.

(i) Find the vector equation of line L that is passing through point A and B.

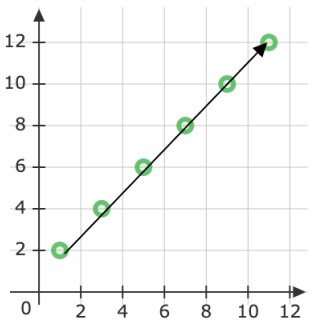
(ii) Sketch the line for $t = 0 : 1 : 5$ and indicate its direction and initial point.

Solution

(i) Direction of vector is parallel with $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 3, 4\rangle - \langle 1, 2\rangle = \langle 2, 2\rangle$
Vector equation of line L is $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v} = \langle 1, 2\rangle + t\langle 2, 2\rangle$

(ii)

t	$\underset{\sim}{r}$
0	$\langle 1, 2\rangle$
1	$\langle 3, 4\rangle$
2	$\langle 5, 6\rangle$
3	$\langle 7, 8\rangle$
4	$\langle 9, 10\rangle$
5	$\langle 11, 12\rangle$



Question 5

If a unit vector $\vec{a}$ makes an angle of $\pi/3$ with $i, \pi/4$ with j and acute angle θ with k, find θ and the components of $\vec{a}$.

Solution Let

$$\alpha, \beta, \gamma$$

to represent angles of

$$\vec{a}$$

, and

$$\vec{a} = a_1 i + a_2 j + a_3 k$$

.

$$\begin{aligned}
\cos\alpha &= \frac{a_1}{|\vec{a}|} \\
\cos\left(\frac{\pi}{3}\right) &= \frac{a_1}{1} \\
a_1 &= \frac{1}{2} \\
\cos\beta &= \frac{a_2}{|\vec{a}|} \\
\cos\left(\frac{\pi}{4}\right) &= \frac{a_2}{1} \\
a_2 &= \frac{1}{\sqrt{2}} \\
\cos\gamma &= \frac{a_3}{|\vec{a}|} \\
\cos\theta &= \frac{a_3}{1} \\
a_3 &= \cos\theta \\
\therefore\ \vec{a} &= \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \cos\theta k \\
|\vec{a}| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (\cos\theta)^2} \\
1 &= \sqrt{\frac{1}{4} + \frac{1}{2} + \cos^2\theta} \\
1 &= \frac{3}{4} + \cos^2\theta \\
\cos^2\theta &= \frac{1}{4} \\
\cos\theta &= \pm\frac{1}{2}
\end{aligned}$$

Since θ is given as an **acute** angle, $\cos\theta > 0$, hence $\cos\theta = \frac{1}{2}$ and $\theta = 60°$.

$$\therefore\ \theta = 60°, \qquad \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \frac{1}{2}k$$

Question 6

If $\vec{a}$ is a unit vector and $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.

Solution Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

$$
\begin{aligned}
(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) &= 8\\
\vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a} &= 8\\
|\vec{x}|^2-|\vec{a}|^2 &= 8\\
|\vec{x}|^2-1 &= 8\\
|\vec{x}|^2 &= 9\\
|\vec{x}| &= 3
\end{aligned}
$$

Note that magnitude of vector is non-negative.

Question 7

Find the gradient for $f = \left(2x^2+y\right) / \left(x^2-y^2\right)$.

Solution

$$
\frac{d}{dx}(f) = \frac{\left(x^2-y^2\right)(4x)-\left(2x^2+y\right)(2x)}{\left(x^2-y^2\right)^2}
$$

$$
\frac{d}{dx}(f) = \frac{2xy(-2y-1)}{\left(x^2-y^2\right)^2}
$$

and

$$
\frac{d}{dy}(f) = \frac{\left(x^2-y^2\right)(1)-\left(2x^2+y\right)(-2y)}{\left(x^2-y^2\right)^2}
$$

$$
\frac{d}{dy}(f) = \frac{x^2+4x^2y+y^2}{\left(x^2-y^2\right)^2}
$$

Hence

$$
\nabla f = \frac{2xy(-2y-1)}{\left(x^2-y^2\right)^2}\boldsymbol{i} + \frac{x^2+4x^2y+y^2}{\left(x^2-y^2\right)^2}\boldsymbol{j}
$$

Question 8

Calculate the divergence of the following vector fields of $F(x,y)$ and $G(x,y)$;

(a) $F = y^3\boldsymbol{i} + xy\boldsymbol{j}$

(b) $G = \frac{4y}{x^2}\boldsymbol{i} + \sin(y)\boldsymbol{j} + 3\boldsymbol{k}$

(c) $G = e^x\boldsymbol{i} + \ln(xy)\boldsymbol{j} + e^{xyz}\boldsymbol{k}$

Solution

(a)

$$
\begin{aligned}
\nabla\cdot\boldsymbol{F}(x,y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\\
&= \frac{\partial}{\partial x}y^3 + \frac{\partial}{\partial y}xy = 0 + x = x
\end{aligned}
$$

(b)

$$
\begin{aligned}
\nabla\cdot\boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\\
&= \frac{\partial}{\partial x}\left(\frac{4y}{x^2}\right) + \frac{\partial}{\partial y}\sin(y) + \frac{\partial}{\partial z}3\\
&= 4y\times\frac{\partial}{\partial x}x^{-2} + \cos(y) = 4y\times(-2)x^{-2-1} + \cos(y)\\
&= -8yx^{-3} + \cos(y) = -\frac{8y}{x^3} + \cos(y)
\end{aligned}
$$

(c)

$$\begin{aligned}\nabla \cdot \boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= \frac{\partial}{\partial x}\mathrm{e}^x + \frac{\partial}{\partial y}\ln(xy) + \frac{\partial}{\partial z}\mathrm{e}^{xyz} \\ &= \mathrm{e}^x + \frac{\partial}{\partial y}(\ln(x) + \ln(y)) + \mathrm{e}^{xyz} \times \frac{\partial}{\partial z}(xyz) \\ &= \mathrm{e}^x + \frac{1}{y} + xy\mathrm{e}^{xyz}\end{aligned}$$

Question 9

Calculate the curl of the following vector fields of $F(x, y, z)$;

(a) $F = 3x^2\boldsymbol{i} + 2z\boldsymbol{j} - x\boldsymbol{k}$

(b) $F = y^3\boldsymbol{i} + xy\boldsymbol{j} - z\boldsymbol{k}$

(c) $F = \left(1 + y + z^2\right)\boldsymbol{i} + (e^{xyz})\,\boldsymbol{j} - (xyz)\boldsymbol{k}$

Solution

(a)

$$\begin{aligned}\nabla \times \boldsymbol{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 2z & -x \end{vmatrix} \\ &= \left(\frac{\partial(-x)}{\partial y} - \frac{\partial(2z)}{\partial z}\right) i - \left(\frac{\partial(-x)}{\partial x} - \frac{\partial\left(3x^2\right)}{\partial z}\right) j + \left(\frac{\partial(2z)}{\partial x} - \frac{\partial\left(3x^2\right)}{\partial y}\right) k \\ &= (0-2)i - (-1-0)j + (0-0)k \\ &= -2i + j\end{aligned}$$

(b)

$$\begin{aligned}\nabla \times \boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \boldsymbol{k} \\ &= \left(\frac{\partial(-z)}{\partial y} - \frac{\partial(xy)}{\partial z}\right) i - \left(\frac{\partial(-z)}{\partial x} - \frac{\partial\left(y^3\right)}{\partial z}\right) j + \left(\frac{\partial(xy)}{\partial x} - \frac{\partial\left(y^3\right)}{\partial y}\right) \boldsymbol{k} \\ &= 0i - 0j + \left(y - 3y^2\right) \boldsymbol{k} \\ &= \left(y - 3y^2\right) \boldsymbol{k}\end{aligned}$$

i.e., the curl vector is in the k direction.

(c)

$$\begin{aligned}\mathrm{curl}(\mathbf{F}) = &\left(\frac{\partial(-xyz)}{\partial y} - \frac{\partial e^{xyz}}{\partial z}\right) \vec{i} \\ &+ \left(\frac{\partial(1 + y + z^2)}{\partial z} - \frac{\partial(-xyz)}{\partial x}\right) \vec{j} \\ &+ \left(\frac{\partial e^{xyz}}{\partial x} - \frac{\partial(1 + y + z^2)}{\partial y}\right) \vec{k}.\end{aligned}$$

$$\mathrm{curl}(\mathbf{F}) = -\left(xz + xye^{xyz}\right) \vec{i} + (2z + yz)\vec{j} + \left(yze^{xyz} - 1\right) \vec{k}$$

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