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Tutorial 4: Vector Algebra II

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Question 1

Two long straight pipes are specified using Cartesian coordinates as follow:

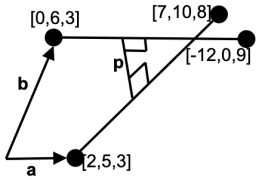
Pipe A: diameter 0.8; axis through points $(2, 5, 3)$ and $(7, 10, 8)$.
Pipe B: diameter 1.0; axis through points $(0, 6, 3)$ and $(-12, 0, 9)$.

Explain if the two pipes require realignment to prevent intersection.

Solution

Pipe A and B have axes:

$$\mathbf{r}_A = [2, 5, 3] + \lambda'[5, 5, 5] = [2, 5, 3] + \lambda\frac{[1, 1, 1]}{\sqrt{3}}$$
$$\mathbf{r}_B = [0, 6, 3] + \mu'[-12, -6, 6] = [0, 6, 3] + \mu\frac{[-2, -1, 1]}{\sqrt{6}}$$



(Non-unit) perpendicular to both axes is

$$\mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = [2, -3, 1]$$

The length of the mutual perpendicular is mod

$$(\mathbf{a} - \mathbf{b}) \cdot \frac{[2, -3, 1]}{\sqrt{14}} = [2, -1, 0] \cdot \frac{[2, -3, 1]}{\sqrt{14}} = 1.87$$

Sum of the radii of the pipes is $0.4 + 0.5 = 0.9$. Hence, the pipes do not intersect.

Question 2

Determine if the sets of vectors given are parallel or non-parallel. Show your answers:

(i) $\underset{\sim}{a} = \langle 2, 4, -1\rangle$ and $\underset{\sim}{b} = \langle -6, -12, 3\rangle$

(ii) $\underset{\sim}{a} = \langle 4, 10\rangle$ and $\underset{\sim}{b} = \langle 2, -9\rangle$

Solution

(i) These two vectors are parallel since $\underset{\sim}{b} = -3\underset{\sim}{a}$

(ii) These two vectors are not parallel since there is no scalar a that can fulfil the scalar multiplication where $\underset{\sim}{a} \neq a\underset{\sim}{b}$

Question 3

Find the unit vectors that are perpendicular to the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as following:

(i) $\underset{\sim}{a} = \langle 2, 4, 5\rangle, \underset{\sim}{b} = \langle 1, 2, -2\rangle$

(ii) $\underset{\sim}{a} = \langle 2, 4, -4\rangle, \underset{\sim}{b} = \langle 1, 2, -2\rangle$

Solution

(i) Vector perpendicular to both vector

$$\underset{\sim}{a}$$

and

$$\underset{\sim}{b}$$

is

$$\underset{\sim}{a} \times \underset{\sim}{b}$$

.

A unit vector perpendicular to both

$$\underset{\sim}{a}$$

and

$$\underset{\sim}{b}$$

is

$$\hat{n} = \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|}$$

.

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 4 & 5 \\ 1 & 2 & -2 \end{vmatrix} = \underset{\sim}{i}(-8-10) - \underset{\sim}{j}(-4-5) + \underset{\sim}{k}(4-4) = \langle -18, 9, 0\rangle$$

$$\hat{n} = \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|} = \frac{\langle -18, 9, 0\rangle}{\sqrt{(-18)^2 + (9)^2}} = \frac{\langle -18, 9, 0\rangle}{9\sqrt{5}} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\rangle$$

Since the question asks for the unit vector**s** (plural), both directions are valid:

$$\pm\hat{n} = \pm\left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$$

(Check: $\langle -2, 1, 0\rangle \cdot \underset{\sim}{a} = -4 + 4 + 0 = 0$ and $\langle -2, 1, 0\rangle \cdot \underset{\sim}{b} = -2 + 2 + 0 = 0$, so it is perpendicular to both.)

The unit vector **exists** in this case.

(ii)

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 4 & -4 \\ 1 & 2 & -2 \end{vmatrix} = \underset{\sim}{i}(-8+8) - \underset{\sim}{j}(-4+4) + \underset{\sim}{k}(4-4) = \langle 0, 0, 0\rangle$$

The vectors are parallel because $a = 2b$. Their zero cross product cannot be normalized, but **infinitely many unit vectors perpendicular to both exist**. A vector $n = (u, v, w)$ is perpendicular to both precisely when $u + 2v - 2w = 0$. For example,

$$n = \frac{1}{\sqrt{5}}\langle -2, 1, 0\rangle, \qquad n \cdot a = n \cdot b = 0, \qquad |n| = 1.$$

More generally, all the required unit vectors are

$$n(\theta) = \cos\theta \frac{\langle -2, 1, 0\rangle}{\sqrt{5}} + \sin\theta \frac{\langle 2, 4, 5\rangle}{3\sqrt{5}}, \qquad 0 \le \theta < 2\pi.$$

The two vectors in this combination are mutually perpendicular unit vectors in the plane perpendicular to b.

Question 4

Find the distance from $P = (-3, 7, 4)$ to the line l with vector equation;

$$r = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

Solution

Here $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$, $A = (2, -2, -3)$ and $\mathbf{u} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$. So, $\overrightarrow{AP}$ represents $\mathbf{v} = \mathbf{p} - \mathbf{a} = \begin{pmatrix} -3 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ 7 \end{pmatrix}$. Now,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} -5 & 3 \\ 9 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 3 \\ -5 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & -5 \\ -5 & 9 \end{vmatrix} \mathbf{k} = \begin{pmatrix} -62 \\ -43 \\ 11 \end{pmatrix}$$

.
Therefore, $|\mathbf{u} \times \mathbf{v}| = \sqrt{(-62)^2 + (-43)^2 + 11^2} = 3\sqrt{646}$ and $|\mathbf{u}| = \sqrt{4^2 + (-5)^2 + 3^2} = 5\sqrt{2}$.

The distance is,

$$\frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}|} = \frac{3\sqrt{646}}{5\sqrt{2}} = \frac{3\sqrt{323}}{5}$$

Note that we have $\mathbf{v} \times \mathbf{u} = (\mathbf{p} - \mathbf{a}) \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ and so $|\mathbf{v} \times \mathbf{u}| = |(\mathbf{p} - \mathbf{a}) \times \mathbf{u}| = |\mathbf{u} \times \mathbf{v}|$.

Question 5

Calculate the distance between the lines l and m having vector equations $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{u}$ and $\boldsymbol{r} = \boldsymbol{b} + \mu \boldsymbol{v}$ respectively, where;

$$\boldsymbol{a} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \boldsymbol{u} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{v} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Solution

We have $\mathbf{b} - \mathbf{a} = 2\mathbf{i} - 5\mathbf{j} + k$ and

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} -3 & 1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -3 \\ -3 & 1 \end{vmatrix} \mathbf{k} = \begin{pmatrix} -4 \\ 4 \\ -8 \end{pmatrix}$$

Thus, we get $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v}) = -8 - 20 - 8 = -36$ and $|\mathbf{u} \times \mathbf{v}| = \sqrt{(-4)^2 + 4^2 + (-8)^2} = 4\sqrt{6}$. The distance from l to m is

$$\frac{|(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|-36|}{\sqrt{96}} = \frac{36}{4\sqrt{6}} = \frac{9}{\sqrt{6}}$$

Question 6

Find the following equation of line for the line L passing through the point $P(3,1,-2)$ and $Q(-2,7,-4)$

(i) Vector equation

(ii) Parametric equation

(iii) Cartesian equation

Solution

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle -2,7,-4\rangle - \langle 3,1,-2\rangle = \langle -5,6,-2\rangle$ is vector parallel to the line L

(i) The vector equation of the line, $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v}$ where $\underset{\sim}{a}$ is position vector of any point at the line and $\underset{\sim}{v}$ is vector parallel to the line.

Thus, $\underset{\sim}{r} = \langle 3,1,-2\rangle + t\langle -5,6,-2\rangle$ is the vector equation of line L

Noted that $\underset{\sim}{r} = \langle -2,7,-4\rangle + t\langle -5,6,-2\rangle$ is also acceptable. To generate consistent data, we always use the first point.

(ii) $\underset{\sim}{r} = \langle x,y,z\rangle$

Compare both side,

$$\begin{aligned} x &= 3-5t \\ y &= 1+6t, \quad t \in \mathbb{R} \\ z &= -2-2t \end{aligned}$$

(iii) Eliminating the parameter t, we get

$$\frac{x-3}{-5} = \frac{y-1}{6} = \frac{z+2}{-2}$$

Question 7

Find the Cartesian equation of plane contains the point $(1,2,-1)$ and perpendicular to the intersecting line of the planes and $2x+y+z=2$ and $x+2y+z=3$.

Solution

The intersecting line has vector that is parallel to both planes, which is the cross product of their respectively normal vector,

$$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \underset{\sim}{i}(1-2) - \underset{\sim}{j}(2-1) + \underset{\sim}{k}(4-1) = \langle -1,-1,3\rangle$$

The Cartesian equation of plane that perpendicular to the intersecting line has the normal vector that is parallel to the vector $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \langle -1,-1,3\rangle$

Therefore, Cartesian equation of plane is $-x-y+3z=k$.

Substitute the point $(1,2,-1)$ which is located on the plane to find k, $k=-1-2+3(-1)=-6$

$$\therefore -x-y+3z=-6$$

Question 8

Find the Cartesian equation of plane contains the line $L_1 : \boldsymbol{r_1} = \boldsymbol{a} + t\boldsymbol{u} = \langle 1, -3, 4\rangle + \langle 2, 1, 1\rangle t$ and parallel to the line $L_2 : \boldsymbol{r}_2 = \boldsymbol{b} + s\boldsymbol{v} = \langle 0, 0, 0\rangle + \langle 1, 2, 3\rangle s$. Then, proof that the plane is parallel to line L_2 ?

Solution
The normal vector of the plane that containing two lines is

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \underset{\sim}{i}(3-2) - \underset{\sim}{j}(6-1) + \underset{\sim}{k}(4-1) = \langle 1, -5, 3\rangle$$

Therefore, Cartesian equation of plane is $x - 5y + 3z = k$. Substitute point $A(1, -3, 4)$ to find k,

$$k = 1 - 5(-3) + 3(4) = 28$$

$$\therefore x - 5y + 3z = 28$$

Note: Point $B(0, 0, 0)$ can't be used to find k because we don't know if it is located on the plane.

To prove that the plane is parallel to line L_2, use dot product where dot product of vector normal and parallel to the plane is equal to zero.

Vector normal to the plane is $\langle 1, -5, 3\rangle$, Vector parallel to the plane is $\langle 1, 2, 3\rangle$.

Since $\langle 1, -5, 3\rangle \cdot \langle 1, 2, 3\rangle = 0$, thus the plane is parallel to line L_2.

Question 9

Find the Cartesian equation of plane contains the line $L_1 : \boldsymbol{r_1} = \boldsymbol{a} + t\boldsymbol{u} = \langle -2, 3, 4\rangle + \langle 1, 2, -1\rangle t$ and line $L_2 : \boldsymbol{r}_2 = \boldsymbol{b} + s\boldsymbol{v} = \langle 3, 4, 0\rangle + \langle -1, -2, 1\rangle s$.

Solution
The plane consists of line L_1 and L_2, therefore point $A(-2, 3, 4)$ and $B(3, 4, 0)$ are located on the plane.

Therefore, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 3, 4, 0\rangle - \langle -2, 3, 4\rangle = \langle 5, 1, -4\rangle$ is one of the line that is parallel to the plane. (Note that a plane is formed by infinite line in various direction, thus there is more than one line parallel to it)

Since the plane contains line L_1 and L_2, thus the plane is parallel to both of them. From the vector equation of line, the plane is parallel $\underset{\sim}{u} = \langle 1, 2, -1\rangle$ and $\underset{\sim}{v} = \langle -1, -2, 1\rangle$ respectively. Note that these two lines are parallel to each other. Thus, they lies in the same plane. Use either one of them to find the plane equation.

Use cross product to find vector normal to the plane,

$$\overrightarrow{AB} \times \underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 5 & 1 & -4 \\ 1 & 2 & -1 \end{vmatrix} = \underset{\sim}{i}(-1+8) - \underset{\sim}{j}(-5+4) + \underset{\sim}{k}(10-1) = \langle 7, 1, 9\rangle$$

Therefore, Cartesian equation is $7x + y + 9z = k$.

Substitute either $A(-2, 3, 4)$ or $B(3, 4, 0)$ to get k,

$$k = 7(-2) + 3 + 9(4) = 25$$

$$\therefore 7x + y + 9z = 25$$

Question 10

Let $\underset{\sim}{a} = \langle 1, -2, -3\rangle$, $\underset{\sim}{b} = \langle 2, 1, -1\rangle$ and $\underset{\sim}{c} = \langle 1, 3, -2\rangle$. Find

(i) $\underset{\sim}{a} \cdot \underset{\sim}{b}(\underset{\sim}{a} \times \underset{\sim}{b})$

(ii) $(\underset{\sim}{a} + \underset{\sim}{b}) \times \underset{\sim}{c}$

Solution

(i)

$$\begin{aligned}
\underset{\sim}{a} \cdot \underset{\sim}{b}\left(\underset{\sim}{a} \times \underset{\sim}{b}\right) &= \langle 1, -2, -3\rangle \cdot \langle 2, 1, -1\rangle(\langle 1, -2, -3\rangle \times \langle 2, 1, -1) \\
&= (2 - 2 + 3)\left(\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix}\right) \\
&= 3\left[\underset{\sim}{i}(2 + 3) - \underset{\sim}{j}(-1 + 6) + \underset{\sim}{k}(1 + 4)\right] \\
&= 3\left(5\underset{\sim}{i} - 5\underset{\sim}{j} + 5\underset{\sim}{k}\right) \\
&= 15\underset{\sim}{i} - 15\underset{\sim}{j} + 15\underset{\sim}{k}
\end{aligned}$$

(ii)

$$\begin{aligned}
(\underset{\sim}{a} + \underset{\sim}{b}) \times \underset{\sim}{c} &= (\langle 1, -2, -3\rangle + \langle 2, 1, -1\rangle) \times \langle 1, 3, -2\rangle \\
&= (\langle 3, -1, -4\rangle) \times \langle 1, 3, -2\rangle \\
&= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -1 & -4 \\ 1 & 3 & -2 \end{vmatrix} \\
&= 14\underset{\sim}{i} + 2\underset{\sim}{j} + 10\underset{\sim}{k}
\end{aligned}$$

Question 11

Determine the shortest distance between the 3D skew lines where;

$$\begin{aligned}
L_a : r_a &= 4\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) \\
L_b : r_b &= \boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})
\end{aligned}$$

Solution

Shortest distance is given as: $\left|(a_a - a_b) \cdot \frac{b_a \times b_b}{|b_a \times b_b|}\right|$

$$\begin{aligned}
L_a : r_a &= 4\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) \\
L_b : r_b &= \boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})
\end{aligned}$$

$$b_a \times b_b = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & -1 \\ 1 & -2 & -1 \end{vmatrix} = (1 - 2)\boldsymbol{i} - (-2 + 1)\boldsymbol{j} + (-4 + 1)\boldsymbol{k} = -\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$$

$$|b_a \times b_b| = \sqrt{(-1)^2 + 1^2 + (-3)^2} = \sqrt{11}$$

$$a_a - a_b = (4 - 1)\boldsymbol{i} + (2 + 3)\boldsymbol{j} + (-6 + 3)\boldsymbol{k} = 3\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$$

Hence, the shortest distance is:

$$\frac{|(3, 5, -3) \cdot (-1, 1, -3)|}{\sqrt{11}} = \frac{|-3 + 5 + 9|}{\sqrt{11}} = \sqrt{11}.$$

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