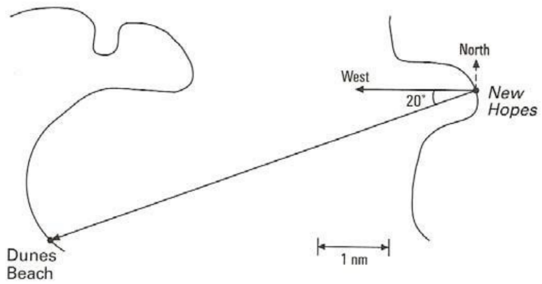
# 5

# Tutorial 5: Engineering Applications of Vector Algebra

Prepared by: Hong Vin | Website: https://kix1001.hongvin.xyz

**Question 1**

The following map shows the location of the docked New Hopes when its skipper decided to navigate to Dunes Beach.




a. What is the heading of the route that the skipper should take to Dunes Beach?

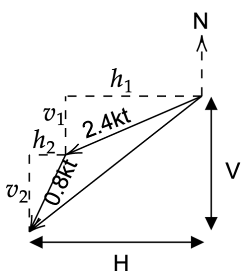
b. Suppose that as the New Hopes heads for Dunes Beach, a strong wind moves the boat at 0.8 knot at a heading of 200°. The skipper sticks to the original heading from Part a despite the wind. Make a labelled sketch of the situation that shows the intended boat path, the effects of the wind and the altered path of the boat.

c. Determine the speed and heading of the boat on its altered path.

**Solution**

a. Heading 250°

b.




c.

$$
\begin{aligned}
h_1 &= 2.4\cos(20^\circ) = 2.255 \\
v_1 &= 2.4\sin(20^\circ) = 0.821 \\
h_2 &= 0.8\sin(20^\circ) = 0.274 \\
v_2 &= 0.8\cos(20^\circ) = 0.752 \\
H &= h_1 + h_2 = 2.529 \\
V &= v_1 + v_2 = 1.573
\end{aligned}
$$

Altered speed $= \sqrt{2.529^2 + 1.573^2} = 2.98$ knots

$$\theta = \tan^{-1}\frac{1.573}{2.529} = 31.88^\circ$$

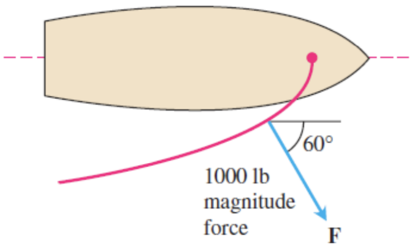
Altered heading$= 180^\circ + (90^\circ - 31.88^\circ) = 238.12^\circ$

**Question 2**

a. It takes 12000 J of work to pull a sled 200 m with a 150 N force. Determine the angle of the rope with the horizontal.

b. Find the work done by a force F = 5i (magnitude 5 N ) in moving an object along the line from the origin to the point $(1, 1)$ (distance in meters).

c. The wind passing over a boat's sail exerted a 1000-lb (pound) magnitude force **F** as shown here. How much work did the wind perform in moving the boat forward 1 mile? Answer in foot-pounds where 1 mile = 5280 foots.




**Solution**

a.

$$\begin{aligned} 12000 &= (150)(200)\cos\theta \\ \theta &= \cos^{-1}\left(\frac{12000}{(150)(200)}\right) \\ &\approx 66^\circ \end{aligned}$$

b.

$$P(0,0), Q(1,1), \mathbf{F} = 5\mathbf{j}$$
$$\overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$$
$$\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = (5\mathbf{j})(\mathbf{i} + \mathbf{j}) = 5\ \mathrm{N} \cdot \mathrm{m} = 5\ \mathrm{J}$$

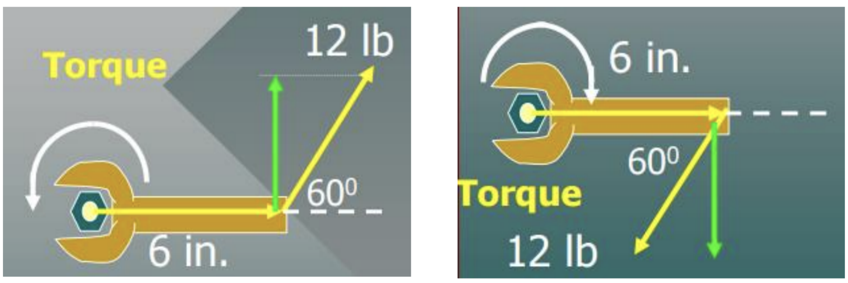
c.

$$\mathbf{W} = |\mathbf{F}|\left|\overrightarrow{PQ}\right|\cos\theta = (1000)(5280)(\cos 60^\circ) = 2,640,000\ \mathrm{ft} \cdot \mathrm{lb}$$

**Question 3**

a. A bolt is tightened using a 20 N force, applied at an angle of 60° to the end of a wrench that is 30 cm long. Calculate the magnitude of the torque about its point of rotation.

b. Explain the difference between both pictures




**Solution**

a.

$$|\vec{\tau}| = (0.3)(20)\sin 60^\circ \approx \underline{\underline{5.2\ \mathrm{J}}}$$

b. Both have the same magnitude but different direction.

**Question 4**

a. Is there a direction $\mathbf{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is $-3°\text{C}/ft$ ?. Give reasons for your answer.

b. A paraboloid of revolution has equation of $2z = x^2 + y^2$. Find the unit normal vector to the surface at the point $(1, 3, 5)$ and normal and tangent line plane to the surface at the same point.

c. Find the equations of the tangent plane and normal line to the surfaces

   i. $2x^2 + y^2 - z^2 = -3$ at $(1, 2, 3)$

   ii. $30 - y^2 - z^2 = x^2$ at $(1, -2, 5)$

**Solution**

a.

$$\nabla T = 2y\mathbf{i} + (2x - z)\mathbf{j} - y\mathbf{k}$$
$$\nabla T(1, -1, 1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$|\nabla T(1, -1, 1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$$

$\Rightarrow$ No. The minimum rate of change is $-\sqrt{6}$.

b.

$$x^2 + y^2 - 2z = 0$$
$$\nabla f(x, y, z) = 2x\hat{\imath} + 2y\hat{\jmath} - 2z\hat{k}$$
$$\nabla f(1, 3, 5) = 2\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$$

Tangent line:

$$2(x - 1) + 6(y - 3) - 2(z - 5) = 0$$
$$2x - 6y - 2z - 10 = 0$$

Normal line:

$$x = 1 + 2t; \quad y = 3 + 6t; \quad z = 5 - 2t \qquad \text{or} \qquad \frac{x-1}{2} = \frac{y-3}{6} = \frac{z-5}{-2} = t$$

c. i.

$$\nabla f(x, y, z) = 4x\hat{\imath} + 2y\hat{\jmath} - 2z\hat{k}$$
$$\nabla f(1, 2, 3) = 4\hat{\imath} + 4\hat{\jmath} - 6\hat{k}$$

Tangent line:

$$4(x - 1) + 4(y - 2) - 6(z - 3) = 0$$
$$4x + 4y - 6z + 6 = 0$$

Normal line:

$$x = 1 + 4t; \quad y = 2 + 4t; \quad z = 3 - 6t$$

ii.

$$x^2 + y^2 + z^2 - 30 = 0$$
$$\nabla f(x, y, z) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$
$$\nabla f(1, -2, 5) = 2\hat{\imath} - 4\hat{\jmath} + 10\hat{k}$$

Tangent line:

$$2(x - 1) - 4(y + 2) + 10(z - 5) = 0$$
$$2x - 4y + 10z - 60 = 0$$

Normal line:

$$x = 1 + 2t; \quad y = -2 - 4t; \quad z = 5 + 10t$$

**Question 5**

a. Determine the divergence of the vector field $F(x,y) = x/yi + (2x - 3y)j$ together with its physical meaning.

b. Determine the curl of the vector field $F(x,y,z) = x\boldsymbol{i} - y\boldsymbol{j} + z\boldsymbol{k}$ together with its physical meaning.

**Solution**

a.

$$\begin{aligned}\nabla \cdot \mathbf{F}(x,y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial}{\partial y}(2x - 3y) \\ &= \frac{1}{y} - 3\end{aligned}$$

The divergence of the vector is a scalar, thus it is either expanding or compressing.

b.

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \\ &= \left(\frac{\partial(z)}{\partial y} - \frac{\partial(-y)}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(z)}{\partial x} - \frac{\partial(x)}{\partial z}\right)\mathbf{j} + \left(\frac{\partial(-y)}{\partial x} - \frac{\partial(x)}{\partial y}\right)\mathbf{k} \\ &= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = 0\end{aligned}$$

Therefore the vector field $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ is an irrotational vector field.

**Question 6**

A force of 2.5 N is applied perpendicular to the handle of a spanner with length of 15 cm to tighten a bolt. Find the torque exerted by the force about the center of the bolt and the direction of the torque. (Ans: $37.5 \times 10^{-2}$Nm )

**Solution**

Angle between $r$ and $F$, $\theta = 90°$

$$\tau = |F||r|\sin\theta = 2.5\text{N} \times 0.015\text{m} \times \sin 90 = 37.5 \times 10^{-2}\text{Nm}$$

Direction: anticlockwise