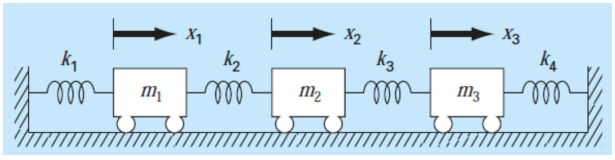
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Tutorial 7: Matrix Algebra for Homogeneous Linear Algebraic System

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Question 1

The following linear algebraic equations represents the 3 dof mass spring systems below:



$$\begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3+k_4}{m_3} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Determine the smallest eigenvalue and the corresponding eigenvector for the eigenvalue/ eigenvector problem if $k_1 = k_4 = 15\frac{N}{m}, k_2 = k_3 = 35\frac{N}{m}$, and $m_1 = m_2 = m_3 = 1.5$ kg. Then, draw the eigenvector.

Solution Put $\lambda = \omega^2$. Substituting the given masses and stiffnesses gives

$$A = \frac{1}{3}\begin{pmatrix} 100 & -70 & 0 \\ -70 & 140 & -70 \\ 0 & -70 & 100 \end{pmatrix}.$$

A nonzero eigenvector requires $\det(A - \lambda I) = 0$. Expanding the determinant,

$$\begin{aligned} \det(A - \lambda I) &= \left(\frac{100}{3} - \lambda\right)\left[\left(\frac{100}{3} - \lambda\right)\left(\frac{140}{3} - \lambda\right) - 2\left(\frac{70}{3}\right)^2\right] \\ &= \left(\frac{100}{3} - \lambda\right)\left(\lambda^2 - 80\lambda + \frac{1400}{3}\right). \end{aligned}$$

Thus, in ascending order,

$$\lambda_1 = 40 - \frac{10\sqrt{102}}{3} \approx 6.3350, \quad \lambda_2 = \frac{100}{3} \approx 33.3333, \quad \lambda_3 = 40 + \frac{10\sqrt{102}}{3} \approx 73.6650.$$

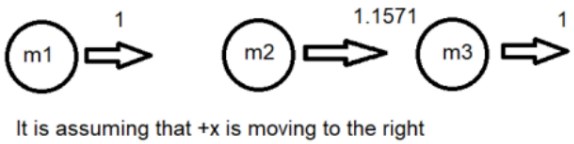
For λ_1, the first and third rows of $(A - \lambda_1 I)x = 0$ give

$$x_2 = \frac{100 - 3\lambda_1}{70}x_1 = \frac{\sqrt{102} - 2}{7}x_1, \qquad x_3 = x_1.$$

Substitution also satisfies the second row. Taking $x_1 = 1$ gives the smallest mode:

$$v_1 = \begin{pmatrix} 1 \\ (\sqrt{102} - 2)/7 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1.1571 \\ 1 \end{pmatrix}.$$

Its eigenspace is $\{tv_1 : t \in \mathbb{R}\}$; eigenvectors require $t \neq 0$. All three masses move in phase, with the middle mass having the largest amplitude.



Question 2

Continue from Q1, determine the second largest eigenvalue and the corresponding normalised eigenvector. Then, draw the eigenvector.

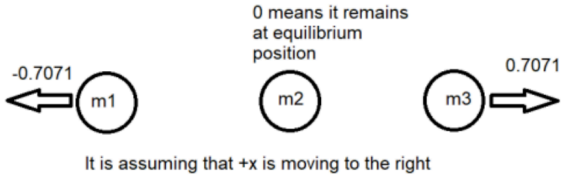
Solution From Q1, the intermediate eigenvalue is $\lambda_2 = 100/3$. Hence

$$A - \lambda_2 I = \frac{1}{3}\begin{pmatrix} 0 & -70 & 0 \\ -70 & 40 & -70 \\ 0 & -70 & 0 \end{pmatrix}.$$

The first row gives $x_2 = 0$; the second then gives $x_1 + x_3 = 0$. Taking $x_3 = 1$ gives $v_2 = (-1, 0, 1)^T$. Its length is $\sqrt{2}$, so

$$\widehat{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \approx \begin{pmatrix} -0.7071 \\ 0 \\ 0.7071 \end{pmatrix}.$$

Either sign is a valid normalized eigenvector. The two outer masses move in opposite directions, while the middle mass is stationary in this mode.



Question 3

Continue from Q1, determine the largest eigenvalue and the corresponding unscaled eigenvector. Then, draw the eigenvector.

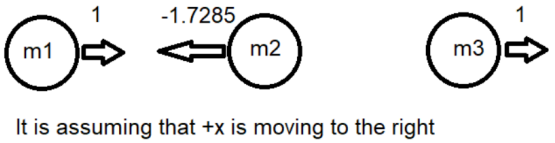
Solution The largest eigenvalue is $\lambda_3 = 40+10\sqrt{102}/3 \approx 73.6650$. The first and third rows of $(A-\lambda_3 I)x = 0$ give

$$x_2 = \frac{100 - 3\lambda_3}{70}x_1 = -\frac{\sqrt{102}+2}{7}x_1, \qquad x_3 = x_1.$$

These values satisfy the second row as well. An unscaled eigenvector is

$$v_3 = \begin{pmatrix} 1 \\ -(\sqrt{102}+2)/7 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 1 \\ -1.7285 \\ 1 \end{pmatrix}.$$

Every nonzero multiple is also an eigenvector. The outer masses move together, in the opposite direction to the middle mass.



Question 4

Combining the results obtained from Q1-Q4, obtain the eigenvector matrix, P and diagonalize the following matrix. Comment the relationship between the diagonal matrix and the eigenvalue.

$$\begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3+k_4}{m_3} \end{bmatrix}$$

Solution Keep the eigenvectors exact before multiplying. Write

$$a = \frac{\sqrt{102}-2}{7}, \qquad b = -\frac{\sqrt{102}+2}{7}, \qquad a-b = \frac{2\sqrt{102}}{7}.$$

Using the eigenvectors from Q1–Q3 as columns,

$$P = \begin{pmatrix} 1 & -1/\sqrt{2} & 1 \\ a & 0 & b \\ 1 & 1/\sqrt{2} & 1 \end{pmatrix}, \qquad D = \operatorname{diag}\left(40 - \frac{10\sqrt{102}}{3}, \frac{100}{3}, 40 + \frac{10\sqrt{102}}{3}\right).$$

To find P^{-1}, solve $P(c_1, c_2, c_3)^T = (x_1, x_2, x_3)^T$. Adding and subtracting its first and third equations gives $c_1 + c_3 = (x_1 + x_3)/2$ and $c_2 = (x_3 - x_1)/\sqrt{2}$. The middle equation $ac_1 + bc_3 = x_2$ then gives

$$P^{-1} = \begin{pmatrix} -\dfrac{b}{2(a-b)} & \dfrac{1}{a-b} & -\dfrac{b}{2(a-b)} \\ -\dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\ \dfrac{a}{2(a-b)} & -\dfrac{1}{a-b} & \dfrac{a}{2(a-b)} \end{pmatrix}.$$

Each column satisfies $Av_i = \lambda_i v_i$, so $AP = PD$ and $P^{-1}AP = D$. The diagonal entries of D are the corresponding eigenvalues, in the same order as the columns of P.

Question 5

Determine $\begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3+k_4}{m_3} \end{bmatrix}^{50}$ and comment on the change of it's eigenvalue and eigenvector, as compared to Q4.

Solution From $A = PDP^{-1}$, adjacent factors $P^{-1}P$ cancel, giving

$$A^{50} = PD^{50}P^{-1}, \qquad D^{50} = \operatorname{diag}(\lambda_1^{50}, \lambda_2^{50}, \lambda_3^{50}).$$

Use the exact values from Q4 before raising them to the 50th power. The result is

$$A^{50} \approx 10^{93} \begin{pmatrix} 0.46255450 & -0.79952578 & 0.46255450 \\ -0.79952578 & 1.38198087 & -0.79952578 \\ 0.46255450 & -0.79952578 & 0.46255450 \end{pmatrix}.$$

The matrix entries shown are rounded; $PD^{50}P^{-1}$ is the exact result. Every eigenvector of A remains an eigenvector of A^{50}, with eigenvalue λ_i^{50}. Here the three eigenvalues are positive and distinct, so taking the 50th power preserves their distinct eigenspaces.

Question 6

Given $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $|\mathbf{B}| = 2$ has an eigenvalue of 2 . Find the remaining eigenvalues and develop the characteristic equation without developing the eigenvalue problem and without performing the determinant.

Solution

$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $|\mathbf{B}| = 2$ has an eigenvalue of 2

$\mathbf{B}$ has 3 eigenvalues since it has 3×3 matrix.

From the property,

$$\begin{aligned}\text{Trace}(\mathbf{B}) &= \sum \lambda_i = \lambda_1 + \lambda_2 + \lambda_3 \\ 1 + 1 + 2 &= \lambda_1 + \lambda_2 + (2) \\ \lambda_1 + \lambda_2 &= 2 \end{aligned} \tag{1}$$

From the property,

$$\begin{aligned}\text{Determinant}(\mathbf{B}) &= \prod \lambda_i = \lambda_1 \lambda_2 \lambda_3 \\ 2 &= \lambda_1 \lambda_2 (2) \\ \lambda_1 \lambda_2 &= 1 \end{aligned} \tag{2}$$

Subs (2) into (1)

$$\begin{aligned}\lambda_1 + \frac{1}{\lambda_1} &= 2 \\ \lambda_1^2 - 2\lambda_1 + 1 &= 0 \\ (\lambda_1 - 1)^2 &= 0 \\ \lambda_1 &= 1 \\ \lambda_2 &= \frac{1}{\lambda_1} = 1 \end{aligned}$$

$\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$

Characteristic equation:

$$\begin{aligned}(\lambda - 1)(\lambda - 1)(\lambda - 2) &= 0 \\ \left(\lambda^2 - 2\lambda + 1\right)(\lambda - 2) &= 0 \\ \lambda^3 + (-4)\lambda^2 + (5)\lambda - 2 &= 0 \end{aligned}$$

Question 7

Continue Q6, using Cayley-Hamilton theorem to verify that:

$$\mathbf{B}^{-1} = \frac{1}{2}\mathbf{B}^2 + (-2)\mathbf{B} + \left(\frac{5}{2}\right)\mathbf{I} \text{ and } \mathbf{B}^6 = \left[(57)\mathbf{B}^2 + (-108)\mathbf{B} + 52\mathbf{I}\right]$$

Then, compute the $\mathbf{B}^5$ via the theorem.

Solution

$$\mathbf{B}^3 + (-4)\mathbf{B}^2 + (5)\mathbf{B} - 2\mathbf{I} = \mathbf{0}$$
$$\mathbf{B}^2 + (-4)\mathbf{B} + (5)\mathbf{I} - 2\mathbf{B}^{-1} = \mathbf{0}$$
$$\mathbf{B}^{-1} = \frac{1}{2}\mathbf{B}^2 + (-2)\mathbf{B} + \left(\frac{5}{2}\right)\mathbf{I} \quad \text{(verified)}$$

$$\mathbf{B}^4 + (-4)\mathbf{B}^3 + (5)\mathbf{B}^2 - 2\mathbf{B} = \mathbf{0}$$
$$\mathbf{B}^4 = (4)\mathbf{B}^3 - (5)\mathbf{B}^2 + 2\mathbf{B}$$
$$\mathbf{B}^4 = \left[(16)\mathbf{B}^2 - (20)\mathbf{B} + 8\mathbf{I}\right] - (5)\mathbf{B}^2 + 2\mathbf{B}$$
$$\mathbf{B}^4 = \left[(11)\mathbf{B}^2 - (18)\mathbf{B} + 8\mathbf{I}\right]$$
$$\mathbf{B}^5 = (11)\mathbf{B}^3 - (18)\mathbf{B}^2 + 8\mathbf{B}$$
$$\mathbf{B}^5 = \left[(44)\mathbf{B}^2 - (55)\mathbf{B} + 22\mathbf{I}\right] - (18)\mathbf{B}^2 + 8\mathbf{B}$$
$$\mathbf{B}^5 = (26)\mathbf{B}^2 - (47)\mathbf{B} + 22\mathbf{I}$$
$$\mathbf{B}^6 = (26)\mathbf{B}^3 - (47)\mathbf{B}^2 + 22\mathbf{B}$$
$$\mathbf{B}^6 = \left[(104)\mathbf{B}^2 + (-130)\mathbf{B} + 52\mathbf{I}\right] - (47)\mathbf{B}^2 + 22\mathbf{B}$$
$$\mathbf{B}^6 = \left[(57)\mathbf{B}^2 + (-108)\mathbf{B} + 52\mathbf{I}\right] \quad \text{(verified)}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{B}^5 = (26)\mathbf{B}^2 - (47)\mathbf{B} + 22\mathbf{I}$$
$$= (26)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - (47)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} + 22\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= (26)\begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix} - (47)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} + 22\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 10 & 145 \\ 0 & 1 & 31 \\ 0 & 0 & 32 \end{bmatrix}$$

Question 8

Explain why the following equation : $\mathbf{B}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$ is not working to solve the $Q7$ problem?

Solution
Previously we obtained,

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$$

where $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

The eigenvalue/eigenvector problem: $(\mathbf{B} - \lambda\mathbf{I})x = \mathbf{0}$

For $\lambda_1 = 1$

$$\begin{bmatrix} 1-1 & 2 & 3 \\ 0 & 1-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xrightarrow[R1 \to R1(1/2)]{} \begin{bmatrix} 0 & 1 & 3/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xrightarrow[R1 \to R1 - R2(3/2)]{R3 \to R3 - R2(1)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note: RREF shows rank 2 (i.e. 2 linearly independent vectors)

$$x_2 = 0$$

$$x_3 = 0$$

Eigenspace for $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=1} = \begin{Bmatrix} x_1 \\ 0 \\ 0 \end{Bmatrix} = t \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\Bigg|_{x_1 = t}$, where $t \in \mathbb{R}$

Note: Unscaled eigenvectors can be any vector of eigenspace. Normally we just let $t = 1$

Eigenvector, $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=1} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

For repeated eigenvalue, $\lambda_2 = 1$

Eigenvector, $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=1} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

For $\lambda_3 = 2$

$$\begin{bmatrix} 1-2 & 2 & 3 \\ 0 & 1-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xrightarrow[R2\to R2(-1)]{R1\to R1(-1)} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xrightarrow{R1\to R1-R2(-2)} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note: RREF shows rank 2 (i.e. 2 linearly independent vectors)

$$x_1 - 5x_3 = 0 \gg x_1 = 5x_3$$

$$x_2 - x_3 = 0 \gg x_2 = x_3$$

Eigenspace for $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=2} = \begin{Bmatrix} 5x_3 \\ x_3 \\ x_3 \end{Bmatrix} = t \begin{Bmatrix} 5 \\ 1 \\ 1 \end{Bmatrix}\Bigg|_{x_3=t}$, where $t \in \mathbb{R}$

Note: Unscaled eigenvectors can be any vector of eigenspace. Normally we just let $t = 1$

Eigenvector, $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=2} = \begin{Bmatrix} 5 \\ 1 \\ 1 \end{Bmatrix}$

Eigenvector or modal matrix, $\mathbf{P} = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(\mathbf{P}) = 1(0) - 1(0) + 5(0) = 0$$

$$\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})} \text{adjoint}(\mathbf{P}) = \frac{\mathbf{1}}{0} \text{adjoint}(\mathbf{P}) = \text{ undefined}$$

Thus, $\mathbf{B}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$ can't be computed.

Question 9

Based on Q7 and Q8, discuss the advantage and disadvantage of diagonalization formula versus the Cayley-Hamilton theorem in solving the power of a matrix.

Solution

	Diagonalization formula for power of a matrix $\mathbf{B}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$	Cayley-Hamilton theorem $f(\mathbf{A}) = p_0\mathbf{I} + p_1\mathbf{A} + p_2\mathbf{A}^2 + \ldots + p_n\mathbf{A}^n = 0$
Advantage	Can compute the power of a matrix much faster with single equation.	The formulation can be developed using characteristic equation only without the extensive calculation of the eigenvector and eigenvalues.
Disadvantage	• Require the complete eigenvalues and eigenvectors data for the diagonal matrix as well as the eigenvector matrix. This procedure can be long if compute manually. • In some cases, it can't be computed because eigenvector matrix might not have inversion especially for repeated root case.	Need derivation of the theorem formula for the computation of the higher power of the matrix.

Question 10

Find all the eigenvalue and normalized eigenvectors in terms of eigenvalue matrix and eigenvector matrix for the matrix $\mathbf{C}$.

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Then, verify the eigenvalue matrix and eigenvector matrix if they satisfy the eigenvalue/eigenvector problem, i.e. $(\mathbf{C} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0}$.

Solution

$$\mathbf{C} - \lambda\mathbf{I} = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

For non-trivial solution, $|\mathbf{C} - \lambda\mathbf{I}| = -\lambda^3 + 3\lambda + 2 = 0$

$$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 2$$

Eigenvalues or spectral matrix, $\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

For $\lambda_1 = -1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xrightarrow[R3 \to R3 - R1]{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note: RREF shows rank 1 (i.e. 1 linearly independent vectors)

$$x_1 + x_2 + x_3 = 0$$

$$x_1 = -x_2 - x_3$$

$$E_{-1} = \left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : t, s \in \mathbb{R} \right\}.$$

Note: Unscaled eigenvectors can be any vector of eigenspace. Normally we just let $t = 1$ or $s = 1$

Eigenvectors, $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=-1} = \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix}$ & $\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$ for repeated eigenvalues $\lambda_1 = -1, \lambda_2 = -1$ respectively.

Note: Normalized eigenvectors has magnitude = 1. It can be obtained by dividing the unscaled eigenvectors with the magnitude.

Normalized eigenvectors, $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=-1} = \begin{Bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix}$ & $\begin{Bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix}$

For $\lambda_2 = 2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_3 = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$$

$$\xrightarrow[R3 \to R3(-1/2)]{R1 \to R1(-1/2) \\ R2 \to R2(-1/2)} \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix} \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_3 = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$$

$$\xrightarrow[R3 \to R3-R1(-1/2)]{R2 \to R2-R1(-1/2)} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 3/4 & -3/4 \\ 0 & -3/4 & 3/4 \end{bmatrix} \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_3 = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$$

$$\xrightarrow[R2 \to R2(4/3)]{} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & -3/4 & 3/4 \end{bmatrix} \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_3 = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$$

$$\xrightarrow[R3 \to R3-R2(-3/4)]{} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_3 = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$$

$$\xrightarrow[R1 \to R1-R2(-1/2)]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_3 = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$$

Note: RREF shows rank 2 (i.e. 2 linearly independent vectors)

$$x_1 - x_3 = 0 \quad \gg x_1 = x_3$$
$$x_2 - x_3 = 0 \quad \gg x_2 = x_3$$

Eigenspace for $\left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_{\lambda=2} = \left\{ \begin{matrix} x_3 \\ x_3 \\ x_3 \end{matrix} \right\} = s \left\{ \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right\} \Bigg|_{x_3=s}$, where $t \in \mathbb{R}$

Note: Unscaled eigenvectors can be any vector of eigenspace. Normally we just let $s = 1$

Eigenvector, $\left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_{\lambda=2} = \left\{ \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right\}$

Note: Normalized eigenvectors has magnitude = 1. It can be obtained by dividing the unscaled eigenvectors with the magnitude.

Normalized Eigenvector, $\left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\}_{\lambda=2} = \left\{ \begin{matrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{matrix} \right\}$

Eigenvector or modal matrix, $\mathbf{P} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

Eigenvalues or spectral matrix, $\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$(\mathbf{C} - \lambda \mathbf{I})\boldsymbol{x} = \mathbf{0}$$
$$\mathbf{C}\boldsymbol{x} = \lambda \boldsymbol{x}$$

$$\mathbf{C}\left[\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_1} \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_2} \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_3}\right] = \left[\lambda_1 \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_1} \quad \lambda_2 \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_2} \quad \lambda_3 \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_3}\right]$$

$$\mathbf{C}\left[\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_1} \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_2} \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_3}\right] = \left[\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_1} \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_2} \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_3}\right] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\mathbf{CP} = \mathbf{PD}$$

$$\text{LHS: } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.7071 & 1.1547 \\ -0.7071 & 0 & 1.1547 \\ 0 & -0.7071 & 1.1547 \end{bmatrix}$$

$$\text{RHS: } \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.7071 & 1.1547 \\ -0.7071 & 0 & 1.1547 \\ 0 & -0.7071 & 1.1547 \end{bmatrix}$$

Since LHS = RHS, the solutions of eigenvalue matrix, **D** & eigenvector matrix, **P** are verified.

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