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Tutorial 8: Integration

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Question 1

Find

$$\int \ln\left(x^2+2\right)dx$$

Solution

Let $u=\ln\left(x^2+2\right)$, $dv=dx$

$$\begin{aligned} u &= \ln\left(x^2+2\right) & dv &= dx \\ du &= \frac{2x}{x^2+2}dx & v &= x \end{aligned}$$

So,

$$\begin{aligned} \int \ln\left(x^2+2\right)dx &= x\ln\left(x^2+2\right) - 2\int \frac{x^2}{x^2+2}dx \\ &= x\ln\left(x^2+2\right) - 2\int\left(1-\frac{2}{x^2+2}\right)dx \\ &= x\ln\left(x^2+2\right) - 2x + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C \\ &= x\left(\ln\left(x^2+2\right)-2\right) + 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C \end{aligned}$$

Question 2

Find

$$\int x^2\ln x dx$$

Solution

Let $u=\ln x$, $dv=x^2\ dx$

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{dx}{x} & v &= \frac{x^3}{3} \end{aligned}$$

So,

$$\begin{aligned} \int x^2\ln x\ dx &= \frac{x^3}{3}\ln x - \int\frac{x^3}{3}\frac{dx}{x} \\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C \end{aligned}$$

Question 3

Find

$$\int x^3 e^{x^2} dx$$

Solution

Let $u = x^2$ and $dv = xe^{x^2} dx$

$$\begin{aligned} u &= x^2 & dv &= xe^{x^2} \\ du &= 2x\ dx & v &= \frac{1}{2}e^{x^2} \end{aligned}$$

Hence,

$$\begin{aligned} \int x^3 e^{x^2}\ dx &= \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2} dx \\ &= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C \\ &= \frac{1}{2}e^{x^2}\left(x^2 - 1\right) + C \end{aligned}$$

Question 4

Find

$$\int \frac{(x+1)}{x^3 + x^2 - 6x} dx$$

Solution

Factoring the denomenator $x^3 + x^2 - 6x = x\left(x^2 + x - 6\right) = x(x-2)(x+3)$. The integrand is now $\frac{(x+1)}{x(x-2)(x+3)}$.

Representing the integrand such that:

$$\frac{(x+1)}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3} \tag{1}$$

Multiplying equation (1) with $x(x-2)(x+3)$,

$$x + 1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2) \tag{2}$$

Let $x = 0$ in (2), $1 = A(-2)(3) + B(0)(3) + C(0)(-2) \ \Rightarrow\ 1 = -6A$. So, $A = -\frac{1}{6}$.

Let $x = 2$ in (2), $3 = A(0)(5) + B(2)(5) + C(2)(0) \ \Rightarrow\ 3 = 10B$. So, $B = \frac{3}{10}$.

Let $x = -3$ in (2), $-2 = A(-5)(0) + B(-3)(0) + C(-3)(-5) \ \Rightarrow\ -2 = 15C$. So, $C = -\frac{2}{15}$.

Therefore,

$$\begin{aligned} \int \frac{(x+1)}{x^3 + x^2 - 6x}\ dx &= \int \left(-\frac{1}{6}\frac{1}{x} + \frac{3}{10}\frac{1}{x-2} - \frac{2}{15}\frac{1}{x+3}\right)\ dx \\ &= -\frac{1}{6}\ln|x| \ + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| \ + C \end{aligned}$$

Question 5

Find

$$\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx$$

Solution

Factoring the denomenator $x^4+3x^2+2=\left(x^2+1\right)\left(x^2+2\right)$.

The integrand is now $\dfrac{x^3+x^2+x+2}{\left(x^2+1\right)\left(x^2+2\right)}$.

Representing the integrand such that:

$$[b]\frac{x^3+x^2+x+2}{\left(x^2+1\right)\left(x^2+2\right)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2} \tag{3}$$

Multiplying equation (3) with $\left(x^2+1\right)\left(x^2+2\right)$,

$$\begin{aligned} x^3+x^2+x+2 &= (Ax+B)\left(x^2+2\right)+(Cx+D)\left(x^2+1\right) \\ &= Ax^3+2Ax+Bx^2+2B+Cx^3+Cx+Dx^2+D \\ &= (A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D) \end{aligned} \tag{4}$$

Comparing LHS and RHS of equation (4),

$$\begin{aligned} A+C&=1 \\ B+D&=1 \\ 2A+C&=1 \\ 2B+D&=2 \end{aligned} \tag{5}$$

Solving equation (5) simultaneously to obtain $A=0,\ B=1,\ C=1,\ D=0$

Therefore,

$$\begin{aligned} \int \frac{x^3+x^2+x+2}{\left(x^2+1\right)\left(x^2+2\right)}dx &= \int\left(\frac{1}{x^2+1}+\frac{x}{x^2+2}\right)dx \\ &= \tan^{-1}x+\frac{1}{2}\ln\left(x^2+2\right)+C \end{aligned}$$

Question 6

Find

$$\int \tan^3(3x)\sec^4(3x)dx$$

Solution

$$\begin{aligned} \int \tan^3(3x)\ \sec^4(3x)\ dx &= \int \tan^3(3x)\ \left(1+\tan^2(3x)\right)\sec^2(3x)\ dx \\ &= \int \tan^3(3x)\sec^2(3x)\ dx+\int \tan^5(3x)\sec^2(3x)\ dx \\ &= \frac{1}{3}\frac{1}{4}\tan^4(3x)+\frac{1}{3}\frac{1}{6}\tan^6(3x)+C \\ &= \frac{1}{12}\tan^4(3x)+\frac{1}{18}\tan^6(3x)+C \end{aligned}$$

Question 7

Find

$$\int \sin^4(x)\cos^7(x)dx$$

Solution

Using trigonometry identity, $\sin^2 x + \cos^2 x = 1 \implies \cos^2 x = 1 - \sin^2 x$

$$\begin{aligned}\int \sin^4(x)\cos^7(x)\ dx &= \int \sin^4(x)\cos^6(x)\cos(x)\ dx \\ &= \int \sin^4(x)\left(1 - sin^2(x)\right)^3 \cos(x)\ dx\end{aligned}$$

Let $u = \sin x$, $du = \cos(x)\ dx$,

$$\begin{aligned}\int \sin^4(x)\cos^7(x)\ dx &= \int u^4\left(1-u^2\right)^3 \quad du \\ &= \int u^4\left(1-u^2\right)\left(1-u^2\right)^2 du \\ &= \int u^4\left(1-u^2\right)\left(1-2u^2+u^4\right) du \\ &= \int u^4\left(1-2u^2+u^4-u^2+2u^4-u^6\right) du \\ &= \int u^4\left(1-3u^2+3u^4-u^6\right) du \\ &= \int u^4 - 3u^6 + 3u^8 - u^{10} du \\ &= \frac{1}{5}u^5 - \frac{3}{7}u^7 + \frac{3}{9}u^9 - \frac{1}{11}u^{11} + C\end{aligned}$$

Substituting back $u = \sin x$,

$$\int \sin^4(x)\cos^7(x)\ dx = \frac{1}{5}\sin^5 x - \frac{3}{7}\sin^7 x + \frac{1}{3}\sin^9 x - \frac{1}{11}\sin^{11} x + C$$

Question 8

Find

$$\int \frac{dx}{x^2\sqrt{9-x^2}}$$

Solution

Let $x = 3\sin\theta$, therefore $\theta = \sin^{-1}\frac{x}{3}$. Then, $dx = 3\cos\theta\ d\theta$, and

$$\begin{aligned}\sqrt{9-x^2} &= \sqrt{9-(3\sin\theta)^2} \\ &= \sqrt{9-9\sin^2\theta} \\ &= 3\sqrt{1-\sin^2\theta} \\ &= 3\sqrt{\cos^2\theta} \\ &= 3\cos\theta\end{aligned}$$

Using defination of inverse sine, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Therefore, $\cos\theta > 0$.

Thus, $\cos\theta = |\cos\theta| = \dfrac{\sqrt{9-x^2}}{3}$.

Hence,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9-x^2}} &= \int \frac{3\cos\theta\ d\theta}{9\sin^2\theta(3\cos\theta)} \\
&= \frac{1}{9}\int \csc^2\theta\ d\theta \\
&= -\frac{1}{9}\cot\theta + C \\
&= -\frac{1}{9}\frac{\cos\theta}{\sin\theta} + C \\
&= -\frac{1}{9}\frac{\frac{\sqrt{9-x^2}}{3}}{\frac{x}{3}} + C \\
&= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C
\end{aligned}$$

Additional Exercises

Please integrate

1. $\displaystyle\int \frac{x+7}{x^2(x+2)}dx$
2. $\displaystyle\int \frac{\cos x}{\sin^3 x + \sin x}dx$

Solution

1. Decompose into partial fractions (There is a repeated linear factor!), getting

$$\int \frac{x+7}{x^2(x+2)}dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}\right) dx$$

After getting a common denominator, adding fractions, and equating numerators, it follows that $Ax(x+2) + B(x+2) + Cx^2 = x+7$;

let $x = 0 : A(0) + B(2) + C(0) = 7 \longrightarrow B = \frac{7}{2}$;

let $x = -2 : A(0) + B(0) + C(4) = 5 \longrightarrow C = \frac{5}{4}$;

let $x = -1 : A(-1) + B(1) + C(1) = -A + \frac{7}{2} + \frac{5}{4} = 6 \longrightarrow A = -\frac{5}{4}$.

$$\begin{aligned}
\int \frac{x+7}{x^2(x+2)}dx &= \int \left(\frac{-5/4}{x} + \frac{7/2}{x^2} + \frac{5/4}{x+2}\right) dx \\
&= \int \left(-(5/4)\frac{1}{x} + (7/2)x^{-2} + (5/4)\frac{1}{x+2}\right) dx \\
&= -(5/4)\ln|x| + (7/2)\frac{x^{-1}}{(-1)} + (5/4)\ln|x+2| + C \\
&= -\frac{5}{4}\ln|x| - \frac{7}{2x} + \frac{5}{4}\ln|x+2| + C
\end{aligned}$$

2. Use the method of u-substitution first. Let $u = \sin x$ so that $du = \cos x dx$.

Substitute into the original problem, replacing all forms of x, getting

$$\begin{aligned}
\int \frac{\cos x}{\sin^3 x + \sin x}dx &= \int \frac{1}{\sin^3 x + \sin x}\cos x dx \\
&= \int \frac{1}{u^3 + u}du
\end{aligned}$$

Factor and decompose into partial fractions.

$$\begin{aligned}
&= \int \frac{1}{u\left(u^2+1\right)} du \\
&= \int \left(\frac{A}{u} + \frac{Bu+C}{u^2+1}\right) du
\end{aligned}$$

After getting a common denominator, adding fractions, and equating numerators, it follows that $A\left(u^2+1\right) + (Bu+C)u = 1$;

let $u = 0 : A(1) + 0 = 1 \longrightarrow A = 1$;

let $u = i : A\left(i^2+1\right) + (Bi+C)i = A(0) + Bi^2 + Ci = -B + Ci = 1$;

it follows that $B = -1$ and $C = 0$.

$$\begin{aligned}
&= \int \left(\frac{1}{u} + \frac{-u}{u^2+1}\right) du \\
&= \ln|u| - \frac{1}{2}\ln\left|u^2+1\right| + C \\
&= \ln|\sin x| - \frac{1}{2}\ln\left|\sin^2 x + 1\right| + C
\end{aligned}$$