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Tutorial 9: Application of integral

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Question 1

Find the area bounded by the curves.

i. $y = x^4 - x^2$ and $y = x^2$ (the part to the right of the y-axis)

ii. $x = y^3$ and $x = y^2$

iii. $y = \cos(\pi x/2)$ and $y = 1 - x^2$ (in the first quadrant)

Solution

(i) We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get,

$$\begin{aligned} f(x) &= g(x) \\ x^4 - x^2 &= x^2 \end{aligned}$$

$x = 0, \sqrt{2}$ (only positive at the right side)
On $(0, \sqrt{2})$ we have $x^2 > \left(x^4 - x^2\right)$, so $g(x) = x^2$ is the **upper** curve. The area is therefore (upper − lower):

$$\begin{aligned} A &= \int_0^{\sqrt{2}} g(x)dx - \int_0^{\sqrt{2}} f(x)dx \\ &= \int_0^{\sqrt{2}} \left(x^2 - \left(x^4 - x^2\right)\right) dx \\ &= \int_0^{\sqrt{2}} \left(2x^2 - x^4\right) dx \\ &= \left[\frac{2x^3}{3} - \frac{x^5}{5}\right]_0^{\sqrt{2}} \\ &= \frac{2 \cdot 2^{\frac{3}{2}}}{3} - \frac{2^{\frac{5}{2}}}{5} = \frac{2^{\frac{5}{2}}}{3} - \frac{2^{\frac{5}{2}}}{5} = \frac{2^{\frac{7}{2}}}{15} \\ &\approx 0.754 \end{aligned}$$

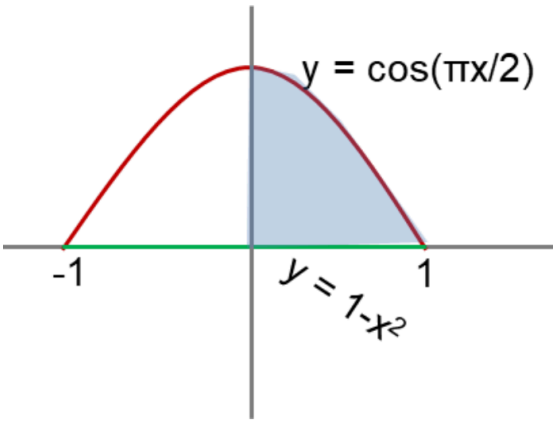
(ii) We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get,

$$\begin{aligned} f(y) &= g(y) \\ y^3 &= y^2 \\ y &= 1, 0 \end{aligned}$$

Now,

$$\begin{aligned} A &= \int_0^1 f(y)dy - \int_0^1 g(y)dy \\ &= \int_0^1 \left(y^2 - y^3\right) dy = \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{aligned}$$

(iii) As from the Figure, the two curves intersect at $(-1, 0)$, $(0, 1)$ and $(1, 0)$. Since we only compute the first quadrant, the limits will be $x = 0$ to $x = 1$.



Now we get,

$$\begin{aligned} \mathrm{A} &= \int_0^1 f(x)dx - \int_0^1 g(x)dx \\ &= \int_0^1 \left(1 - x^2 - \cos\frac{\pi x}{2}\right) dx \\ &= \left| x - \frac{x^3}{3} - \frac{2}{\pi}\sin\frac{\pi x}{2} \right|_0^1 \\ &= \frac{2}{3} - \frac{2}{\pi} \approx 0.03 \text{ Unit} \end{aligned}$$

Question 2

Find the area bounded by the parabola $x = 8 + 2y - y^2$, the y-axis, and the lines $y = -1$ and $y = 3$.

Solution

$$\begin{aligned} x &= 8 + 2y - y^2 \\ &= -\left(y^2 - 2y - 8\right) \\ &= -\left((y-1)^2 - 9\right) \\ &= (4 - y)(2 + y) \end{aligned}$$

$\therefore$ The vortex of the parabola is $(9, 1)$ and cuts the y-axis at $y = -4$, $y = -2$.
Area bounded by the parabola:

$$\begin{aligned} \int_{-1}^{3} \left(8 + 2y - y^2\right)\ dy &= 8y + y^2 - \frac{1}{3}y^3 \Big|_{-1}^{3} \\ &= \left(8(3) + 3^2 - \frac{1}{3}(3)^3\right) - \left(8(-1) + (-1)^2 - \frac{1}{3}(-1)^3\right) \\ &= \frac{92}{3} \end{aligned}$$

Question 3

Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m as shown Figure 3.1. How much work is needed to empty a tank partially filled with water?

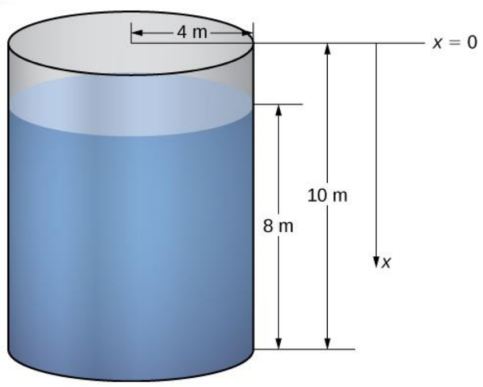


Figure 3.1 : A tank partially filled with water

Solution

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (Figure 3.1).

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2, 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[2, 10]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x \times i \in [x_i - 1, x_i]$. Figure 3.2 shows a representative layer.

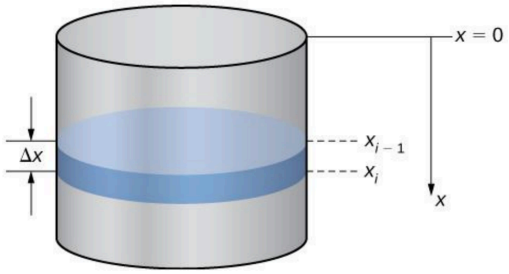


Figure 3.2: A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is 9800N/m^3, or 62.4lb/ft^3, calculating the volume of each layer gives us the weight. In this case, we have:

$$V = \pi(4)^2\Delta x = 16\pi\Delta x$$

Then, the force needed to lift each layer is,

$$F = 9800 \cdot 16\pi\Delta x = 156,800\pi\Delta x$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the i-th layer of water W_i is approximately:

$$W_i \approx 156,800\pi x_i^*\Delta x$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 156,800\pi x_i^*\Delta x$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get,

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x \\ &= 156,800\pi \int_2^{10} x dx \\ &= 156,800\pi \left(\frac{x^2}{2}\right)\Big|_2^{10} \\ &= 7,526,400\pi \approx 23,644,883 \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

Question 4

An object moves along a straight line with acceleration given by $a(t) = -\cos(t)$, and $s(0) = 1$ and $v(0) = 0$. Find the maximum distance the object travels from zero, and find its maximum speed. Describe the motion of the object.

Solution We compute:

$$\begin{aligned} v(t) &= v_0 + \int_0^t -\cos(u) du \\ &= 0 - \sin(u)|_0^t \\ &= -\sin t \end{aligned}$$

The position is then:

$$s(t) = s(0) + \int_0^t -\sin u \, du = 1 + [\cos u]_0^t = 1 + \cos t - 1 = \cos t$$

- **Maximum distance from zero:** $\max |s(t)| = \max |\cos t| = 1$ (attained at $t = 0, \pi, 2\pi, \dots$).
- **Maximum speed:** $\max |v(t)| = \max |-\sin t| = 1$ m/s (attained at $t = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$).

Motion: the object oscillates back and forth between $s = -1$ and $s = 1$ in simple harmonic motion (period 2π), starting at $s = 1$ with zero velocity.

Question 5

A particle at rest leaves the origin with its velocity increasing with time according to v(t) = 3.2tm/s. At 5.0 s, the particle's velocity starts decreasing according to [16.0 − 1.5(t − 5.0)]m/s. This decrease continues until t = 11.0 s, after which the particle's velocity remains constant at 7.0 m/s. (a) What is the acceleration of the particle as a function of time? (b) What is the position of the particle at t = 2.0 s, t = 7.0 s, and t = 12.0 s ?

Solution

(a)

$$
\begin{aligned}
a(t) &= 3.2 \text{ m/s}^2 && t \le 5.0 \text{ s} \\
a(t) &= -1.5 \text{ m/s}^2 && 5.0 \text{ s} \le t \le 11.0 \text{ s} \\
a(t) &= 0 \text{ m/s}^2 && t > 11.0 \text{ s} \\
x(t) &= \int v(t)dt + C_2 \\
&= \int 3.2tdt + C_2 = 1.6t^2 + C_2 && t \le 5.0 \text{ s} \\
x(0) &= 0 \Rightarrow C_2 = 0 && \text{therefore, } x(2.0 \text{ s}) = 6.4 \text{ m} \\
x(t) &= \int v(t)dt + C_2 \\
&= \int [16.0 - 1.5(t - 5.0)]dt + C_2 \\
&= 16t - 1.5\left(\frac{t^2}{2} - 5.0t\right) + C_2 && 5.0 \le t \le 11.0 \text{ s} \\
x(5 \text{ s}) &= 1.6(5.0)^2 = 40 \text{ m} \\
&= 16(5.0 \text{ s}) - 1.5\left(\frac{5^2}{2} - 5.0(5.0)\right) + C_2
\end{aligned}
$$

(b)

$$
\begin{aligned}
40 &= 98.75 + C_2 \Rightarrow C_2 = -58.75 \\
x(7.0 \text{ s}) &= 16(7.0) - 1.5\left(\frac{7^2}{2} - 5.0(7)\right) - 58.75 = 69 \text{ m} \\
x(t) &= \int 7.0dt + C_2 = 7t + C_2 \qquad t \ge 11.0 \text{ s} \\
x(11.0 \text{ s}) &= 16(11) - 1.5\left(\frac{11^2}{2} - 5.0(11)\right) - 58.75 = 109 = 7(11.0 \text{ s}) + C_2 \Rightarrow C_2 = 32 \text{ m} \\
x(t) &= 7t + 32 \text{ m} \\
x &\ge 11.0 \text{ s} \Rightarrow x(12.0 \text{ s}) = 7(12) + 32 = 116 \text{ m}
\end{aligned}
$$

Question 6

A force of 50 N will stretch a spring from its natural length of 2 meters to a length of 2.5 meters.

i. Find the spring constant.

ii. Find the work needed to stretch the spring by a length of 0.5 meter from its natural length.

Solution

(i) we find that the spring constant k is,

$$
\begin{aligned}
k(0.5 \text{ m}) &= 50 \text{ N} \\
k &= 100\text{N/m}
\end{aligned}
$$

(ii) By Hooke's Law, the force needed to compress the spring by a length of x feet from its natural length is

$$f(x) = kx = 100x$$

since $k = 100$.

As the spring expand from a length of 2m to a length of 2.5m, we can think of one end of the spring as moving along the x-axis from $x = 0$ to $x = 0.5$ while the other end of the spring is held fixed. Then the work done is,

$$W = \int_0^{0.5} 100x\,dx = 100 \cdot \frac{0.5^2}{2} = 12.5 \text{ Joule}$$

Question 7

Find the arc length L of the curve $y = x3/2$ from $x = 0$ to $x = 5$.

Solution

$$y' = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$\begin{aligned}
L &= \int_0^5 \sqrt{1+(y')^2}\, dx \\
&= \int_0^5 \sqrt{1+\left(\frac{3}{2}\sqrt{x}\right)^2}\, dx \\
&= \int_0^5 \sqrt{1+\frac{9}{4}x}\, dx \\
&= \int_0^5 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}}\, dx \\
&= \left.\frac{\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\left(\frac{9}{4}\right)}\right|_0^5 \\
&= \frac{\left(1+\frac{9}{4}(5)\right)^{\frac{3}{2}}}{\frac{27}{8}} - \frac{\left(1+\frac{9}{4}(0)\right)^{\frac{3}{2}}}{\frac{27}{8}} \\
&= \frac{335}{27} \\
&= \frac{\left(1+\frac{9}{4}(5)\right)^{\frac{3}{2}}}{\frac{27}{8}} - \frac{\left(1+\frac{9}{4}(0)\right)^{\frac{3}{2}}}{\frac{27}{8}} \\
&= \frac{335}{27}
\end{aligned}$$

Question 8

Find the arc length of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$ from $x = 0$ to $x = a$.

Solution

$$\begin{aligned}
y' &= \frac{a}{2}\left(\frac{1}{a}e^{\frac{x}{a}} - \frac{1}{a}e^{-\frac{x}{a}}\right) + 0\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \\
&= \frac{a}{2}\left(\frac{1}{a}\right)\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \\
&= \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)
\end{aligned}$$

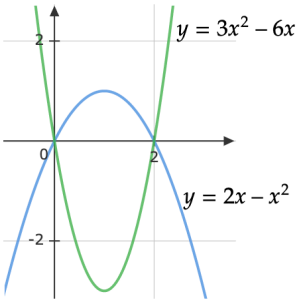
Arc length:

$$
\begin{aligned}
L &= \int_0^a \sqrt{1+(y')^2}\, dx \\
&= \int_0^a \sqrt{1+\left[\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)\right]^2} dx \\
&= \int_0^a \sqrt{1+\frac{1}{4}\left(e^{\frac{2x}{a}}-2e^0+e^{-\frac{2x}{a}}\right)} dx \\
&= \int_0^a \sqrt{1+\frac{1}{4}e^{\frac{2x}{a}}-\frac{1}{2}+\frac{1}{4}e^{-\frac{2x}{a}}}\, dx \\
&= \int_0^a \sqrt{\frac{1}{4}e^{\frac{2x}{a}}+\frac{1}{2}+\frac{1}{4}e^{-\frac{2x}{a}}}\, dx \\
&= \int_0^a \sqrt{\frac{1}{4}\left[e^{\frac{2x}{a}}+2+e^{-\frac{2x}{a}}\right]} dx \\
&= \int_0^a \sqrt{\frac{1}{4}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2} dx \\
&= \int_0^a \frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)\, dx \\
&= \frac{1}{2}\int_0^a \left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)\, dx \\
&= \frac{1}{2}\left[\frac{e^{\frac{x}{a}}}{\frac{1}{a}}-\frac{e^{-\frac{x}{a}}}{\frac{1}{a}}\Bigg|_0^a\right] \\
&= \frac{1}{2}\cdot a\left[e^1-e^{-1}-1+1\right] \\
&= \frac{a}{2}\left[e-e^{-1}\right]
\end{aligned}
$$

Question 9

Find the centroid of the plane area bounded by the parabolas $y = 2x - x^2$ and $y = 3x^2 - 6x$

Solution



$$\begin{aligned}
A &= \int\int_R dA \\
&= \int_0^2 \int_{3x^2-6x}^{2x-x^2} dy dx \\
&= \int_0^2 \left[2x - x^2 - 3x^2 + 6x\right] dx \\
&= \int_0^2 \left[-4x^2 + 8x\right] dx \\
&= \frac{-4x^3}{3} + 4x^2 \Big|_0^2 \\
&= \frac{-4(2)^3}{3} + 4(2)^2 \\
&= \frac{16}{3}
\end{aligned}$$

$$\begin{aligned}
M_y &= \int\int_R x \; dA \\
&= \int_0^2 \int_{3x^2-6x}^{2x-x^2} x \; dy dx \\
&= \int_0^2 x\left(2x - x^2 - 3x^2 + 6x\right) \; dx \\
&= \int_0^2 \left[-4x^3 + 8x^2\right] dx \\
&= -x^4 + \frac{8}{3}x^3 \Big|_0^2 \\
&= -2^4 + \frac{8}{3}(2)^3 \\
&= \frac{16}{3}
\end{aligned}$$

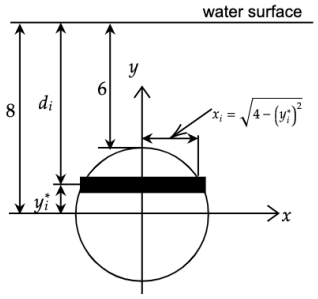
$$\begin{aligned}
M_x &= \int\int_R y \; dA \\
&= \int_0^2 \int_{3x^2-6x}^{2x-x^2} y \; dy dx \\
&= \frac{1}{2}\int_0^2 \left[\left(2x - x^2\right)^2 - \left(3x^2 - 6x\right)^2\right] dx \\
&= \frac{1}{2}\int_0^2 \left[4x^2 - 4x^3 + x^4 - 9x^4 + 36x^3 - 36x^2\right] dx \\
&= \frac{1}{2}\int_0^2 \left[-8x^4 + 32x^3 - 32x^2\right] dx \\
&= \frac{1}{2}\left[-\frac{8}{5}x^5 + 8x^4 - \frac{32}{3}x^3 \Big|_0^2\right] \\
&= \frac{1}{2}\left[-\frac{8}{5}(2)^5 + 8(2)^4 - \frac{32}{3}(2)^3\right] = -\frac{64}{15}
\end{aligned}$$

Hence, $x = \dfrac{M_y}{A} = \dfrac{\frac{16}{3}}{\frac{16}{3}} = 1$, $y = \dfrac{M_x}{A} = \dfrac{-\frac{64}{15}}{\frac{16}{3}} = -\dfrac{4}{5}$. Thus, the centroid is $\left(1, -\frac{4}{5}\right)$.

Question 10

Find the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water.

Solution



Assume that the top of the circular plate is 6 meters under the water.
Setting up the axis system such that the origin of the axis is at the center of the plate.
Finally, split up the plate into n horizontal strips each of width Δy. Choosing a point y_i^* from each strip. Assume that the strips are rectangular.
The depth below the water surface of each strip is,

$$d_i = 8 - y_i^*$$

and that in turn gives us the pressure on the strip,

$$P_i = \rho g d_i = 9810\,(8 - y_i^*)$$

The area of each strip is,

$$A_i = 2\sqrt{4 - (y_i^*)^2}\ \ \Delta y$$

The hydrostatic force on each strip is,

$$F_i = P_i A_i = 9810\,(8 - y_i^*)\,(2)\sqrt{4 - (y_i^*)^2}\ \ \Delta y$$

The total force on the plate is,

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} 19620\,(8 - y_i^*)\,\sqrt{4 - (y_i^*)^2}\ \ \Delta y \\ &= 19620 \int_{-2}^{2} (8 - y)\sqrt{4 - y^2}\ dy \\ &= 19620 \int_{-2}^{2} 8\sqrt{4 - y^2}\ dy - 19620 \int_{-2}^{2} y\sqrt{4 - y^2}\ dy \end{aligned}$$

The first integral requires the trig substitution $y = 2\sin\theta$ and the second integral needs the substitution $v = 4 - y^2$.

$$\begin{aligned} F &= 627840 \int_{-\pi/2}^{\pi/2} \cos^2\theta\ d\theta + 9810 \int_{0}^{0} \sqrt{v}\ dv \\ &= 313920 \int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta)\ d\theta + 0 \\ &= 313920 \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{-\pi/2}^{\pi/2} \\ &= 313920\pi \end{aligned}$$

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