WEEK 11: MULTIPLE INTEGRALS IN POLAR COORDINATE & ITS ENGINEERING APPLICATION

11.1 CARTESIAN COORDINATES TO POLAR COORDINATES

We consider the borders of the region R in terms of x and y and integrate by that. It is how we learned to deal with integrating multivariate functions over a variety of different types of regions in the XY-plane. There will be times when it is far more practical to think about the region R in polar rather than Cartesian coordinates. Consider what would occur, for instance, if we tried to integrate the function f (x, y) across the region R below in Cartesian coordinates:

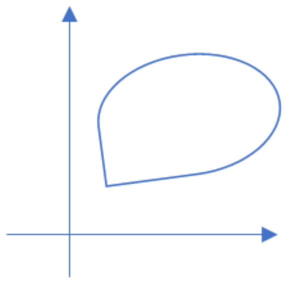


Figure 11.1

It would be challenging to set up the integral in Cartesian coordinates because the region's bounds are neither stated in terms of functions of x nor of y. Instead, we would need to divide the region into smaller sections and build an integral for each. The issue, however, would be made simpler by writing up the integral in terms of **polar coordinates**. Substantially, we will discover how to evaluate specific integrals using polar coordinates in this part.

Let's review some fundamentals of polar coordinates before delving into the method's specifics.

The Cartesian coordinates (x, y), where x and y are measured along the respective axes, can describe any point on the plane. Points on the plane can also be thought of in terms of the polar coordinates r and θ, so this is not the only way to represent them.

Fixing a point O, the origin, and an initial ray will let us construct the polar coordinate system (which generally corresponds to the positive part of the x-axis). Using the directed angle θ from the original ray to the segment OP and the directed distance r from the origin, we can characterize a point P in the plane as follows:

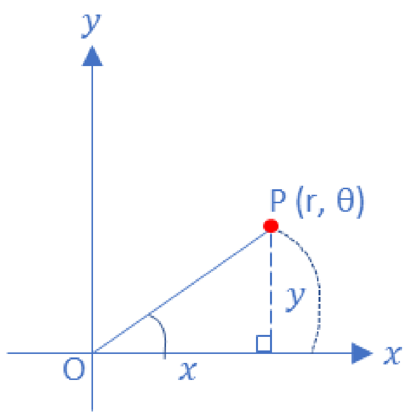


Figure 11.2

If we wish to convert a point's polar coordinates to Cartesian coordinates, or vice-versa, we can use a basic trigonometry to help us out. Recall that, if (x, y) is the Cartesian coordinate of a point with angle θ from the initial ray, and if $x^2 + y^2 = r^2$, then **sin θ =** $\frac{y}{r}$ **and cos θ =** $\frac{x}{r}$**:**

So, if P has polar coordinates (r, θ), then we can rewrite the coordinates using the conversions x = r cos θ and y = r sin θ. Alternatively, if we have Cartesian coordinates (x, y), then we can determine r and θ using the formulas $x^2 + y^2 = r^2$, and tan θ =$\frac{y}{x}$. Summary of cartesian coordinates to polar coordinates:

Cartesian coordinates	Polar coordinates
x	r cos θ
y	r sin θ
f (x, y)	f (r cos θ, r sin θ)

11.2 DOUBLE INTEGRAL IN POLAR COORDINATE & ITS APPLICATION

If we convert rectangular coordinates to polar coordinates, it can often be considerably simpler to evaluate double integrals. But first, we need to define the idea of a double integral in a polar rectangular region before we explain how to execute this change.

When we defined the double integral for a continuous function in rectangular coordinates—say, *g* over a region R in the XY-plane—we divided R into sub-rectangles with sides parallel to the coordinate axes. These sides have either constant x -values and/or constant y -values.

In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r -values and/or constant θ -values. This means we can describe a polar rectangle as in Figure 11.4, with R= {(r, θ) a ≤ r ≤ b, α ≤ θ ≤ β}.

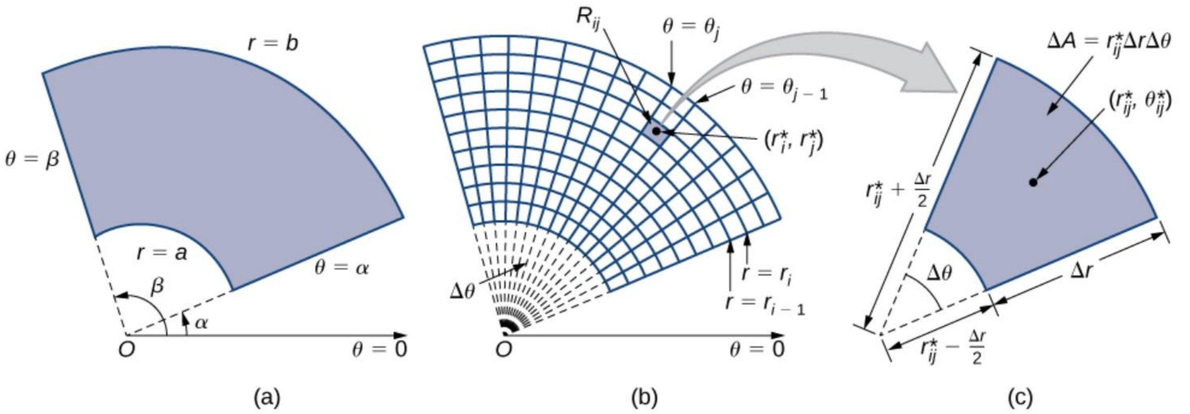


Figure 11.4 (a) A polar rectangle *R* (b) divided into sub rectangles R_{ij} (c) Close-up of a sub-rectangle

Consider a function f (r, ϑ) over a polar rectangle R. We divide the interval $[a,b]$ into m subintervals $[r_{i-1},r_i]$ of length $\Delta r = (b-a)/m$ and divide the interval $[\alpha,\beta]$ into n subintervals $[\vartheta_{i-1},\vartheta_i]$ of width $\Delta\theta = (\beta-\alpha)/n$. This means that the circles $r=r_i$ and rays $\vartheta=\vartheta_i$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar sub-rectangles R_{ij} (Figure 11.4b).

As previously, we must determine the "polar" volume of the thin box above R_{ij} and the area, *dA*, of the polar sub-rectangle R_{ij}. Remember that in a circle with radius *r*, the length *s* of an arc under the influence of a central angle of θ radians is equal to $s=r\vartheta$. As you can see, the polar rectangle R_{ij} resembles a trapezoid with parallel sides $r_{i-1}\Delta\vartheta$ and $r_i\Delta\vartheta$ and a width of Δr. Therefore, the polar sub rectangle R_{ij}'s area is:

$$\Delta A = 12\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta)$$

Simplifying and letting:

$$r_{ij}^* = 12(r_{i-j} + r_i)$$

we have $\Delta A = r_{ij}^*\Delta r\Delta\theta$

Hence, the thin box above R_{ij}'s polar volume (Figure 11.5) is

$$f(r_{ij}^*\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta$$

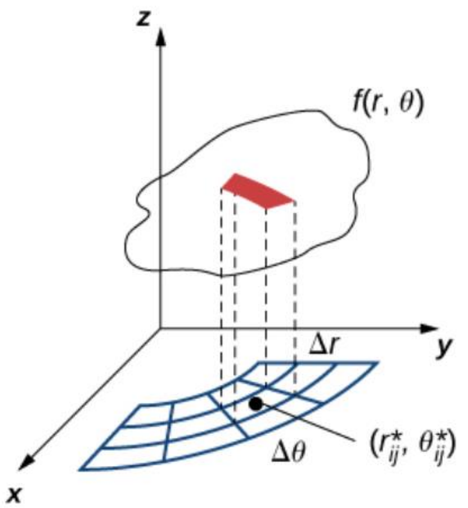


Figure 11.5 Volume of the thin box above polar rectangle, R_{ij}

We obtain a double Riemann sum by applying the same approach to all the sub rectangles and summing the volumes of the rectangular boxes.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*}) r_{ij}^{*} \Delta r \Delta\theta$$

As we have previously seen, as we allow m and n to grow greater, we get a better approximation to the polar volume of the solid above the region R. Consequently, we define the polar volume as the double Riemann sum's limit.

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*}) r_{ij}^{*} \Delta r \Delta\theta$$

This becomes the equation for the double integral.

The following is the definition of the double integral of the function *f(r, ϑ)* over the polar rectangular region R in the *r*- θ plane:

$$\iint R\ f(r,\theta)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*}) r_{ij}^{*} \Delta A$$

$$\lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*}) r_{ij}^{*} \Delta r \Delta\theta$$

The double integral over a polar rectangular region can be stated as an iterated integral in polar coordinates, like the section on double integrals over rectangular regions. Hence,

$$\iint R\ f(r,\theta)\ dA = \iint R\ f(r,\theta)\ rdrd\theta = \int_{\theta=\propto}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)\ rdrd\theta$$

Observe that when using polar coordinates, the expression for *dA* is changed to *rdrd*θ. The polar double integral can also be viewed by substituting the double integral in rectangular coordinates. When the function f is expressed in terms of *x* and *y*, *x=rcos, y=rsin, and dA=r drd*θ, it becomes

$$\iint R\ f(r,\theta)\ dA = \iint R\ f(rcos\theta, rsin\theta)\ rdrd\theta$$

11.2.1 Double Integral in Polar Coordinate & Its Application

Example 11.1

$\iint \boldsymbol{R}\ \boldsymbol{xy}\ \boldsymbol{dA}$ over the region Q, bound by $\boldsymbol{x^2+y^2=4, x=y}$ and $\boldsymbol{x=0}$

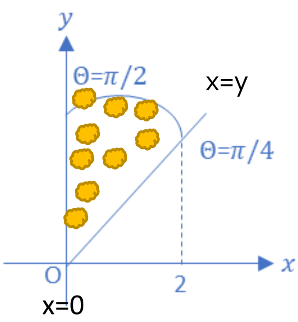
Steps:

1. Always draw the region first, $x^2+y^2 = 4$ is equation for circle,

$$x^2+y^2 = 4$$
$$x^2+y^2 = r^2$$

   Hence, r = 2

2. x= y and x = 0.



3. Set up the integral

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2} xy\ dA = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2} rcos\theta rsin\theta\, rdrd\theta$$

4. Solve the double integral

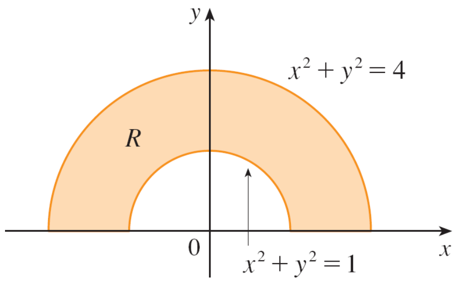
$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2} r^3\ cos\theta sin\theta\ drd\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2} \frac{r^3 sin2\theta}{2} drd\theta$$

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{r^4}{4}\frac{sin2\theta}{2}\Big|_{r\,=\,0}^{r\,=\,2} d\theta = \frac{1}{4}\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} 16\frac{sin2\theta}{2}\ d\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} 2sin2\theta\ \ d\theta = \left(-\frac{cos2\theta}{2}\right)_{\theta\,=\,\frac{\pi}{4}}^{\theta\,=\,\frac{\pi}{2}} = 1$$

Example 11.2

Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

1. Draw the region: It is the half-ring and in polar coordinates it is given by 1 ⍰ *r* ⍰ 2, 0 ⍰ ⍰ ⍰ ⍰.



2. Set up the integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} (3x + 4y^2)dA = \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} (3rcos\theta + 4(rsin\theta)^2)rdrd\theta$$

3. Solve the double integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} (3r^2cos\theta + 4r^3sin^2\theta)\ drd\theta \ \ = \frac{15\pi}{2}$$

11.2.2 Evaluating a Double Integral Over a General Polar Region

In this part, we consider two types of regions, which are comparable to Type I and Type II as stated for rectangular coordinates in section on Double Integrals over General Regions, to calculate the double integral of a continuous function by iterated integrals over general polar regions. We define a general polar region as *r=f(ϑ)* than *ϑ=f(r),* so we describe a general polar region as *R= {(r, ϑ) α ≤ ϑ ≤β, h_1(ϑ) ≤ r ≤h_2(ϑ)}*

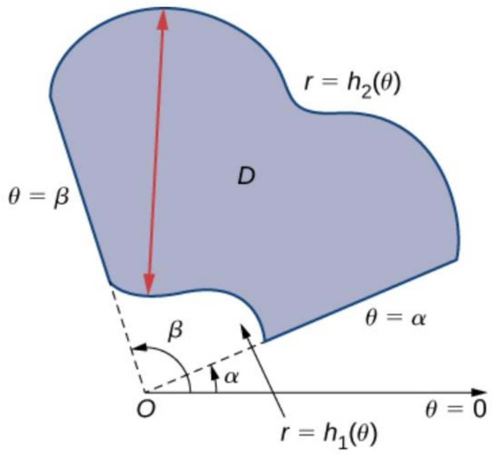


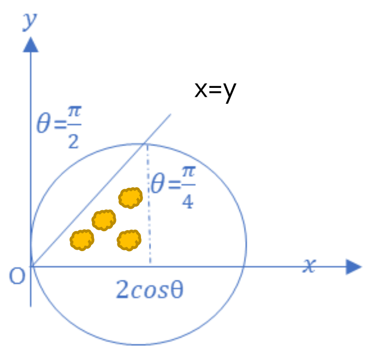
Figure 11.6 A general polar region between *α ≤ ϑ ≤β, h_1(ϑ) ≤ r ≤h_2(ϑ)}*

If *f(r, ϑ)* is continuous on a general polar region D as described above, then

$$\int_{\theta=\propto}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, rdrd\theta$$

Example 11.3

$\iint R\ y\ dA$ over the region Q, bound by $x^2 + y^2 = 2x$ and $x = y$



$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = 2x$$
$$r^2 = 2rcos\theta$$
$$r = 2cos\theta$$

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2cos\theta} rsin\theta \, dA = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2cos\theta} rsin\theta \, rdrd\theta$$

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2cos\theta} r^2 sin\theta \, drd\theta$$

Use integration by part to solve the double integral, you'll get

$$\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2cos\theta} r^2 sin\theta \, drd\theta = \frac{1}{6}$$

11.2.3 Application of Double Integral Using Polar Coordinates to Find Volume

Example 11.4

Find the volume between region $x^2 + y^2 + z^2$= 2 and region z=$\sqrt{x^2 + y^2}$

Solution:

Both regions have the equation of sphere, find the intersection

$x^2 + y^2$+z^2= 2 ------ (1)

z=$\sqrt{x^2 + y^2}$ --------------- (2)

(1) − (2)

$x^2 + y^2 = 1$

Hence, r = 1

Find which region is at top and bottom by plug in (0,0) into eq. (1) and eq. (2)

z = $x^2 + y^2$+z^2= 2 ------ (1) → z = 2 (top)

z=$\sqrt{x^2 + y^2}$ --------------- (2) → z = 0 (bottom)

To find the volume between two regions, we need to know the $z_{between}$

Hence, $z_{between}$ = z_{top} -z_{bottom} → $\sqrt{2 - x^2 - y^2}$ - $\sqrt{x^2 + y^2}$ = $\sqrt{2 - r^2}$ - $\sqrt{r^2}$ = $\sqrt{2 - r^2} - r$

Set up the integral:

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (\sqrt{2 - r^2} - r)\, r dr d\theta$$

Solve the integral:

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r\sqrt{2 - r^2} - r^2\, dr d\theta = \frac{4\pi}{3}\sqrt{2} - 1$$

11.3 TRIPLE INTEGRAL IN CYLINDRICAL COORDINATE AND SPHERICAL COORDINATE & ITS APPLICATION

11.3.1 Polar Coordinates Versus Spherical Coordinates

To handle issues requiring circular symmetry more easily, we previously showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates. Similar circumstances arise with triple integrals. However, in this case, it is important to distinguish between spherical and cylindrical symmetry. This section transforms the triple integrals in rectangular coordinates into a triple integral in cylindrical or spherical coordinates.

As we have previously seen, a point with rectangular coordinates (x, y) in two-dimensional space R-2 can be converted to polar coordinates (r cos ϑ, r sin ϑ) and vice versa. The relationships between the variables are as follows: x=r cos ϑ, y=r sin ϑ, r^2=x^2+y^2, and tan=(y/x).

A point with rectangular coordinates (*x, y, z*) in three-dimensional space R-3 can be identified with cylindrical coordinates (r, ϑ, z), and vice versa. The vertical distance to the point from the *xy*- plane, added as *z*, can be calculated using the same conversion relationships.

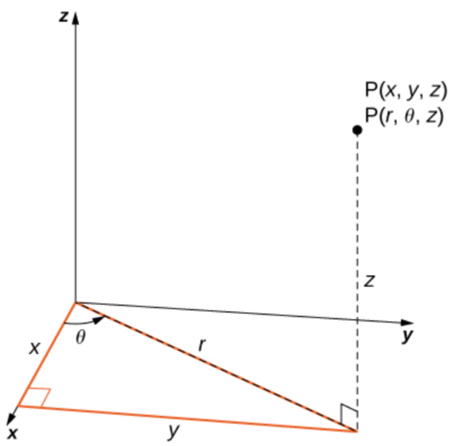


Figure 11.7 Cylindrical coordinates are identical to polar coordinates with vertical z-coordinate as addition.

Notes:

Cylindrical coordinates are polar coordinates with a ‘z’ component.

Polar coordinates *(r, ϑ)* → cylindrical coordinates *(r, ϑ, z)*

The ‘r’ is the distance to projection point on the xy-plane.

The ‘θ’ is the angle from the +ve x-axis to the projection point on the xy-plane.

The ‘z’ is the height from the projection point to the xy-plane.

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

11.3.2 Triple Integral in Cylindrical Coordinates

When evaluating triple integrals, cylindrical coordinates are frequently easier to use than rectangular ones. The following list in Table 11.1 includes several typical surface equations in rectangular coordinates and their corresponding equations in cylindrical coordinates.

Table 11.1 list of typical surface equation

	Cylinder	**Cone**	**Sphere**	**Paraboloid**
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylinder	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

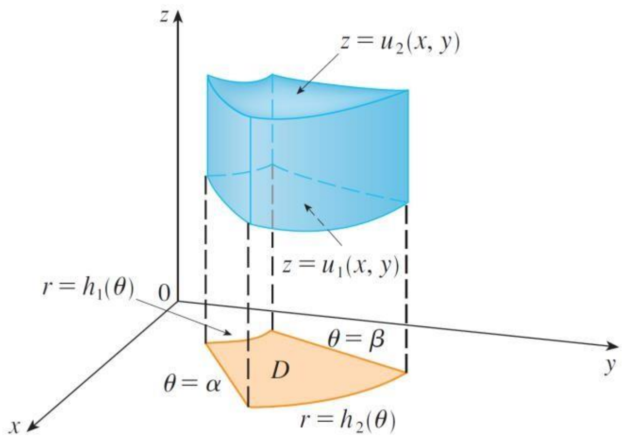
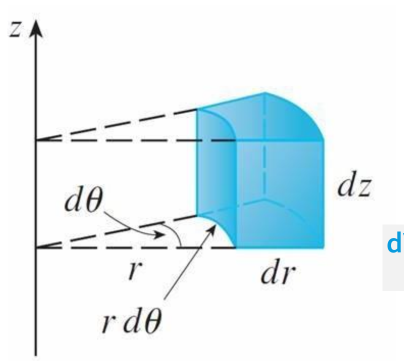


Figure 11.8 Type I region

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 11.8). It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r \cos \vartheta$, $y = r \sin \vartheta$, leaving z as it is, using the appropriate limits of integration for z, r, and ϑ, and replacing dV by $r\, dz\, dr\, d\vartheta$. (Figure 11.9 shows how to remember this.



dV= $r\, dz\, dr\, d\vartheta$

Figure 11.9: Volume element in cylindrical coordinates: *(r, ϑ, z)*

Suppose that *f* is continuous and

$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$

where *D* is given in polar coordinates by

$D = \{(r, \vartheta) \mid \propto \leq \theta \leq \beta , h_1(\vartheta) \leq r \leq h_2(\vartheta)\}$

We know

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Hence, to evaluate the triple integral for cylindrical coordinates, we use the following formula:

$$\iiint \boldsymbol{E\, f(x, y, z)\, dzrdrd\theta} = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=d}^{r=c} \int_{z=a}^{z=b} \boldsymbol{f(r\cos\theta, r\sin\theta, z)\, dzrdrd\theta}$$

It is worthwhile to use this formula when *E* is a solid region easily described in cylindrical coordinates, and when function *f* (*x*, *y*, *z*) involves the expression $x^2 + y^2$.

11.3.2.1 Finding A Cylindrical Volume Using Triple Integral

Example 11.5

Find the volume of T: solid bound by $x^2 + y^2 + z^2 = 9$ and $8z = x^2 + y^2$

Steps:

If possible, always solve the *dz* first as we will end up with *r dr dϑ* (which like double integral).

$x^2 + y^2 + z^2 = 9$ ---- (1)

$8z = x^2 + y^2$---- (2)

To determine which Z region is top and bottom of plane is by plug in $(0,0)$ into the eq. (1) and (2).

We'll get z= 3 --- eq. (1) and z= 0 --- eq. (2). Thus, eq. (1) at top and eq. (2) at bottom.

$x^2 + y^2 + z^2 = 9$ ---- (1) → top

$r^2 + z^2 = 9$

$z = \sqrt{9 - r^2}$

$8z = x^2 + y^2$---- (2) →bottom

$8z = r^2$

$z = r^2/8$

$$\frac{r^2}{8} \leq z \leq \sqrt{9 - r^2}$$

Then, we find r by finding the intersection.

eq. (1)– eq. (2)

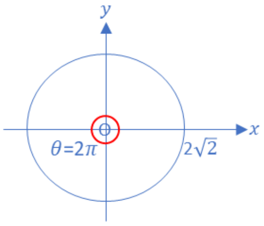
$z^2 + 8z - 9 = 0$

z= -9 and 1 (we only consider the +ve value)

When z=1

$8\,(1) = x^2 + y^2$

$x^2 + y^2 = r^2$ → r = $2\sqrt{2}$



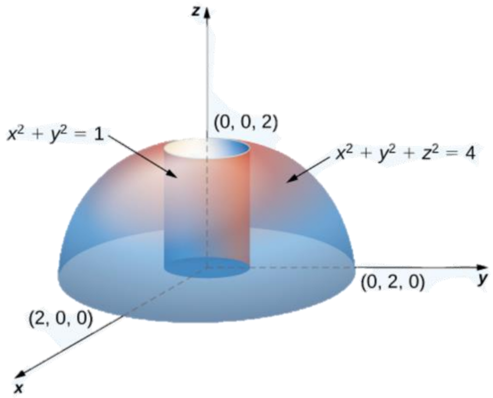
Set up the integral:

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2\sqrt{2}} \int_{z=\frac{x^2+y^2}{8}}^{z=\sqrt{9-x^2-y^2}} 1 \;\; dz\, r dr d\theta$$

Answer: $40\frac{\pi}{3}$

Example 11.6

Let E be the region bounded below by the rθ -plane, above by the sphere $x^2+y^2+z^2=4$, and on the sides by the cylinder $x^2+y^2=1$. Set up a triple integral in cylindrical coordinates to find the volume of the region



Solution:

Solve the dz first

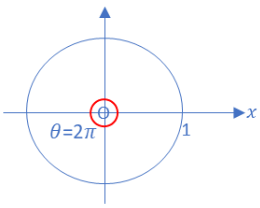
Use the equation of sphere to find z:

$x^2+y^2+z^2=4 \rightarrow z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$

Hence, $0 \leq z \leq \sqrt{4-r^2}$

The equation of cylinder: $x^2+y^2=1^2 \rightarrow x^2+y^2=r^2$, $r = 1$

Hence, $0 \leq r \leq 1$

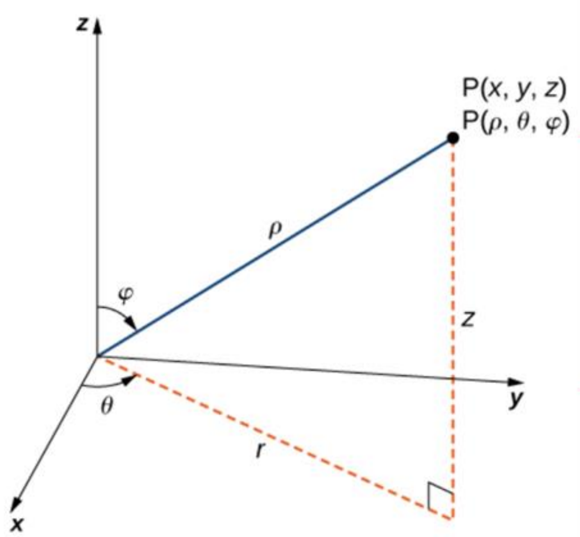


Set up the integral:

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=\sqrt{4-r^2}} 1\ dz\, r dr d\theta = 2\pi\left(\frac{8}{3}-\sqrt{3}\right)$$

11.3.3 Triple Integral in Spherical Coordinates

In three-dimensional space R-3 in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle ϕ from the positive z-axis and the line OP.



Because this is **spherical coordinate**, we must translate in terms of **ρ, ϕ, $\boldsymbol{\theta}$**.

Where,

$sin\varphi = \frac{r}{\rho}, r = \rho \sin\varphi$

Thus.

$x = r\ cos\theta \rightarrow \rho \sin\varphi\ \ cos\theta$

$y = r\ sin\theta \rightarrow \rho \sin\varphi\ \ sin\theta$

What about z?

Figure 11.10 The spherical coordinate system locates points with two angles and a distance from the origin.

* Spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones.

Notes:

The definition for r and θ is the same as cylindrical coordinates. However, for spherical coordinates there are two additional symbols (ρ and ϕ).

It is very important to remember

cylindrical coordinate: (r, $\boldsymbol{\theta}$, z)

spherical coordinate: (ρ, $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$)
(ρ $\geq, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$)

Cylindrical equation: $x^2 + y^2 = r^2$

Spherical equation: $x^2 + y^2 + z^2 = \rho^2$

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. For the volume element of the subbox ΔV in spherical coordinates, we have ΔV=(Δρ)(ρΔϕ)(ρsinϕΔθ), as shown in the following Figure 11.11.

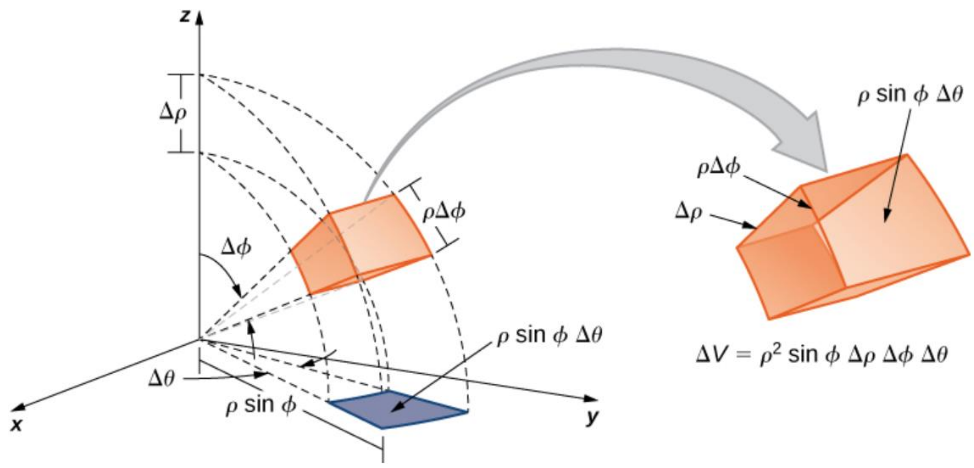


Figure 11.11 The volume element of a box in spherical coordinates

We know

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

Hence, to evaluate the triple integral for cylindrical coordinates, we use the following formula:

$x = r\,cos\theta \rightarrow \rho \sin\varphi\ cos\theta$, $y = r\,sin\theta \rightarrow \rho \sin\varphi\ sin\theta$, and $r = \rho \sin\varphi$,
Where, spherical coordinate: (ρ, θ, φ) and dV= $\rho^2 sin\varphi\ d\rho d\theta d\varphi$

$$\iiint E\, f(x, y, z)\, dV = \iiint T\, f(\rho, \theta, \varphi)\, \rho^2 sin\varphi \ \ d\rho d\varphi d\theta$$

$$\iiint \boldsymbol{T\, f(\rho, \theta, \varphi)\, \rho^2 sin\varphi \ \ d\rho d\varphi d\theta} = \int_{\theta=\alpha}^{\theta=\beta} \int_{\varphi=\varphi_1}^{\varphi=\varphi_1} \int_{\rho=a}^{\rho=b} \boldsymbol{f(\rho, \theta, \varphi)\ \rho^2 sin\varphi \ \ d\rho d\varphi d\theta}$$

11.3.3.1 Finding A Spherical Volume Using Triple Integral

Example 11.7

$\iiint_T \sqrt{x^2 + y^2 + z^2}\, dV$, **The region 'T' is sphere with equation of** $x^2 + y^2 + z^2 = 1$

Solution:

For spherical coordinates, always solve $d\rho$ first.

$$x^2 + y^2 + z^2 = \rho^2$$
$$x^2 + y^2 + z^2 = 1$$

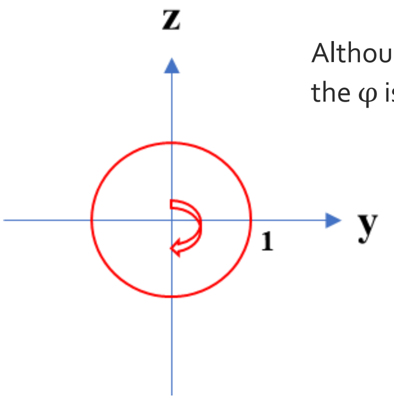
Hence, $\rho = \pm 1$, we only consider positive value

Remember: ρ ≥, $0 \le \theta \le 2\pi, 0 \le \varphi \le \pi$)

Then, solve $d\varphi$, set x= o, Remember, φ always on yz-plane. Then, we will get

$$0 + y^2 + z^2 = 1$$

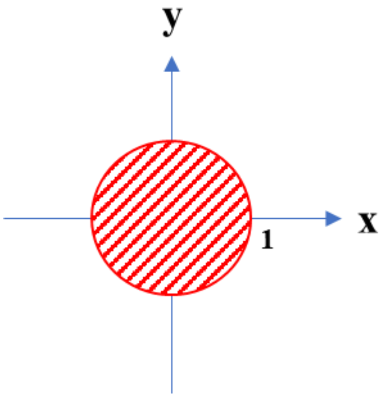
$$0 \le \varphi \le \pi$$



Although it can form 2π,
the φ is never more than π

Next, solve $d\theta$ by setting z=0. Remember θ always on xy-plane → we will get $y^2 + x^2 = 1$

$$0 \le \theta \le 2\pi$$



Set up the integral: $\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi}\int_{\rho=0}^{\rho=1}\sqrt{x^2+y^2+z^2}\,\rho^2 sin\varphi d\rho d\varphi d\theta$

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi}\int_{\rho=0}^{\rho=1}\sqrt{\rho^2}\,\rho^2 sin\varphi d\rho d\varphi d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi}\int_{\rho=0}^{\rho=1}\rho^3 sin\varphi d\rho d\varphi d\theta = \pi$$

Example 11.8

$\iiint T\, xz\, dV$, The region 'T' is solid bound by $x^2+y^2+z^2=4$ and $z=\sqrt{x^2+y^2}$

Solution:

Observe the given equation if the question did not stated type of geometrical shape.

In this question:

$x^2+y^2+z^2=4$ → spherical shape

$z=\sqrt{x^2+y^2}$ → cone shape

For spherical coordinates, always solve $d\rho$ first.

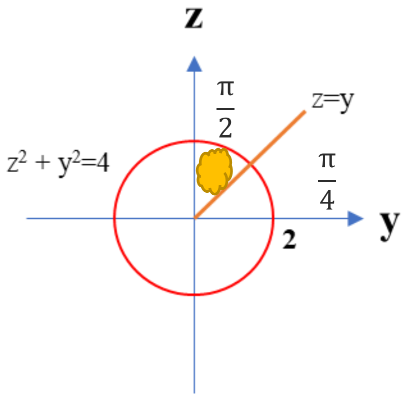
$x^2+y^2+z^2=\rho^2$

$x^2+y^2+z^2=4$

$\rho=2$

$0\le\rho\le 2$

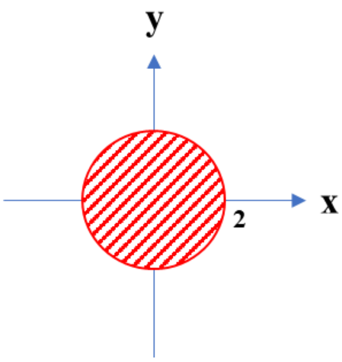
Then, to solve $d\varphi$, set x= o, we will get $y^2+z^2=4$ (from the spherical, it gives us circle equation) and $z=\sqrt{y^2}$ (from the cone, it gives us line equation)



Although it can form π, the geometry for the region 'T' is ice cream cone shape which involved only the upper half cylinder and a cone. Hence, the φ is:

$$\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$$

Next, solve $d\theta$ by setting z=0. Remember θ always on xy-plane → we will get $\boldsymbol{x^2+y^2=4}$



$$0 \leq \theta \leq 2\pi$$

Set up the integral:

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=\frac{\pi}{4}}^{\varphi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=2} xz\ \rho^2 sin\varphi d\rho d\varphi d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=\frac{\pi}{4}}^{\varphi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=2} \rho sin\varphi cos\theta \rho cos\varphi\ \rho^2 sin\varphi d\rho d\varphi d\theta$$

Fo this equation, you will end up with

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=\frac{\pi}{2}}^{\varphi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=2} \rho sin\varphi cos\theta \rho cos\varphi\ \rho^2 sin\varphi d\rho d\varphi d\theta = 0$$

Tips: if you end up with zero for polar coordinate question, try to change the $\int_{\theta=0}^{\theta=2\pi}$ $to\ 2\int_{\theta=0}^{\theta=\pi}$ $OR\ 4\int_{\theta=0}^{\theta=\frac{\pi}{2}}$ $OR\ etc.$

11.3.4 Application of Triple Integral Using Polar Coordinates to Find

11.3.4.1 Centre of Mass

The expressions for the centre of mass $(\bar{x}, \bar{y}, \bar{z})$ of a solid of density $\rho(x, y, z)$ are given below

$$\bar{x} = \frac{\int \rho(x,y,z)x\,dV}{\int \rho(x,y,z)\,dV} = \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{\int \rho(x,y,z)y\,dV}{\int \rho(x,y,z)\,dV} = \frac{M_{xz}}{M}$$

$$\bar{z} = \frac{\int \rho(x,y,z)z\,dV}{\int \rho(x,y,z)\,dV} = \frac{M_{xy}}{M}$$

Where

$$M = \iiint T\,\rho(x,y,z)dV = \iiint T\,\rho(x,y,z)\;dzrdrd\theta$$

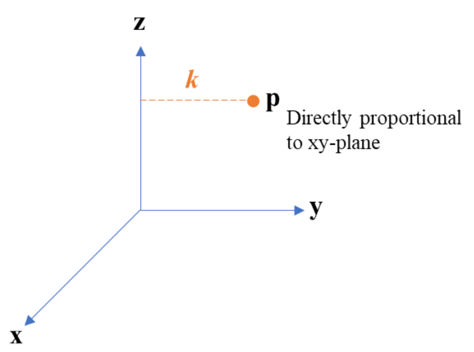
If ρ does not vary with position, these simplify to

$$\bar{x} = \frac{\int x\,dV}{\int dV} \qquad \bar{y} = \frac{\int y\,dV}{\int dV} \qquad \bar{z} = \frac{\int z\,dV}{\int dV}$$

Example 11.9

Find centre of mass of solid bound by $x^2 + y^2 = 4$, $z = 0$, $z = 3$, where the mass density at a point is directly proportional to the point distance from xy-plane.

Solution:



Mass density,
$\rho(x, y, z) = k.z$

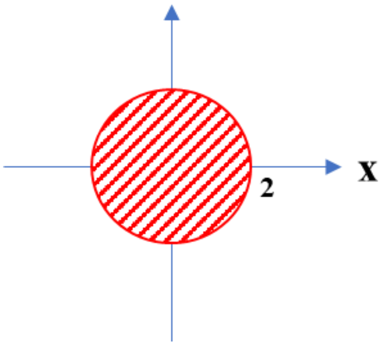
$x^2 + y^2 = 4$ is a circle equation and the geometrical shape is cylindrical

Hence, we need to find r, θ, z

The dz has been solved where question gave us z= 0 and z =3

Then, find r,

$$x^2 + y^2 = 4, r = 2$$



$$0 \leq r \leq 2$$

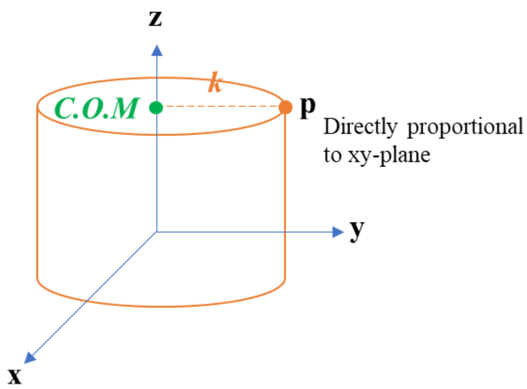
$$0 \leq \theta \leq 2\pi$$

Set up the integral

$$M = \iiint T\, \rho(x, y, z) dV = \iiint T\, \rho(x, y, z)\; dzrdrd\theta$$

$$M = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=3} k.z\, dz\, r\, drd\theta = 18\, k\pi$$

In this question, the centre of mass for the mass density at a point is directly proportional to the distance from xy-plane. Hence, the centre of mass for $\bar{x}$ *and* $\bar{y}$ is 0. Thus, we only need to find the $\bar{z}$.



$$M_{xy} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=3} z.k.z\, dz\, r\, drd\theta = 36\, k\pi$$

$$\bar{z} = \frac{36k\pi}{18k\pi} = 2$$

The centre of mass (C.O.M) = (0,0,2)

11.3.4.2 Moment of Inertia

Recall back section 10.4.2 (moment of inertia)

The moment of inertia I of a small particle of mass m is defined as

$$I = \text{Mass} \times \text{Distance}^2 \text{ or } I = md^2$$

where d is the perpendicular distance from the particle to the axis.

To find the Moment of Inertia of a larger object, it is necessary to carry out a volume integration over all such particles. The distance of a particle at (x, y, z) from the z-axis is given by $\sqrt{x^2 + y^2}$ so the moment of inertia of an object about the z-axis is given by

$$I_z = \int_V \rho(x, y, z)\,(x^2 + y^2) dV$$

Similarly, the Moments of Inertia about the x- and y-axes are given by

$$I_x = \int_V \rho(x, y, z)\,(z^2 + y^2)\, dV$$

$$I_y = \int_V \rho(x, y, z)\,(x^2 + z^2)\, dV$$

Example 11.10

Find the moment of inertia of a uniform sphere of mass *M* and radius *a* about a diameter.

Solution:

A sphere of radius a has volume $4\pi a^3/3$, so that its density is $3M/4\pi a^3$. Then the moment of inertia of the sphere about the z axis is

$$I = \frac{3M}{4\pi a^3} \iiint_V (x^2 + y^2)\, dx\, dy\, dz$$

In this example it is natural to use spherical polar coordinates (recall that $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\, dy\, dz = r^2 \sin\theta\ dr\, d\theta\, d\phi$), so that

$$I = \frac{3M}{4\pi a^3} \iiint_V (r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \cos^2\phi) r^2 \sin\theta\, d\phi\, d\theta\, dr$$

$$I = \frac{3M}{4\pi a^3} \iiint_V (r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi) r^2 \sin\theta\, d\phi\, d\theta\, dr$$

$$= \frac{3M}{4\pi a^3} \iiint_V (r^2 \sin^2 \theta)\, r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \frac{3M}{4\pi a^3} \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^4 \sin^3 \theta \, d\phi \, d\theta \, dr$$

$$= \frac{3M}{4\pi a^3} \int_{r=0}^{a} r^4 dr \int_{\theta=0}^{\pi} \sin^3 \theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{3M}{4\pi a^3} \left[\frac{1}{5} r^5\right]_{r=0}^{a} \left[\frac{1}{12} \cos 3\theta + \frac{3}{4} \cos\theta\right]_{\theta=0}^{\pi} [\phi]_{\phi=0}^{2\pi}$$

$$= \frac{3M}{4\pi a^3} \left[\frac{1}{5} r^5\right]_{r=0}^{a} \left[\frac{1}{12} \cos 3\theta - \frac{3}{4} \cos\theta\right]_{\theta=0}^{\pi} [\phi]_{\phi=0}^{2\pi}$$

$$= \frac{3M}{4\pi a^3} \left(\frac{8}{15} \pi a^5\right)$$

$$= \frac{2}{5} M a^2. \quad \blacksquare$$