STOKES' THEOREM

WEEK 14: STOKES' THEOREM

14.1 INTRODUCTION

From previous topic, we have learnt that double integrals over a region in the plane can be transformed into line integrals over the boundary curve of the region and conversely. We shall now see that more generally, surface integrals over a surface S with boundary curve C can be transformed into line integrals over C and conversely.

Stokes' theorem is the 'curl analogue' of the divergence theorem and related the integral of curl of a vector field over an open surface *S* to the line integral of the vector field around the perimeter *C* bounding the surface.

If **F** is a vector filed existing over surface *S* and around its boundary, closed curve *c*, then

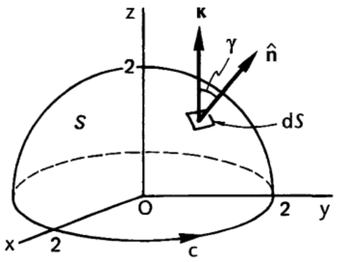
$$\int_S \text{curl } \mathbf{F}.d\mathbf{S} = \oint_c \mathbf{F}.d\mathbf{r}$$

This means that we can express a surface integral in terms of a line integral round the boundary curve.

Example 14.1:

A hemisphere *S* is defined by $x^2 + y^2 + z^2 = 4$ (z ≥ 0). A vector field F=2yi-xj+xzk exists over the surface and around its boundary *c*.

Verify Stokes' theorem, that $\int_S \text{curl } \mathbf{F}.d\mathbf{S} = \oint_c \mathbf{F}.d\mathbf{r}$



$S: x^2 + y^2 + z^2 - 4 = 0$
$F = 2y\boldsymbol{i} - x\boldsymbol{j} + xz\boldsymbol{k}$
c is the circle $x^2 + y^2 = 4$

(a) $\oint_c F.dr = \int_c (2y\boldsymbol{i} - x\boldsymbol{j} + xz\boldsymbol{k}).(\boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz)$

$$= \int_c (2ydx - xdy + xzdz)$$

Converting to polar coordinates

x= 2cosθ; y= 2sinθ; z=0

dx= -2sinθ dθ; dy=2cosθdθ; limits θ= 0 to 2π

Making the substitution and completing the integral

$$\oint_c \boldsymbol{F}.\boldsymbol{dr} = \int_0^{2\pi} (\boldsymbol{4}\ \boldsymbol{sin\theta}[-\boldsymbol{2sin\theta d\theta}] - \boldsymbol{2cos\theta 2cos\theta d\theta})$$

$$= -4\int_0^{2\pi}\left(2\sin^2\theta + \cos^2\theta\right)\,d\theta$$

$$= -4\int_0^{2\pi}\left(1+\sin^2\theta\right)\,d\theta \;=\; -2\int_0^{2\pi}\left(3-\cos 2\theta\right)d\theta \qquad \textbf{(1)}$$

$$= -2\left[3\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -12\pi$$

(b) Now we determine $\int_s curl\;\mathbf{F}.d\mathbf{S}$

$$\int_s curl\;\mathbf{F}.d\mathbf{S} = \int curl\;\mathbf{F}.\hat{\mathbf{n}}\quad dS$$

$$curl\,\mathrm{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -x & xz \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(z-0) + \mathbf{k}(-1-2) = -z\mathbf{j} - 3\mathbf{k}$$

Now $$n = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{2}$$

Then $$\int_s curl\,\mathbf{F}.\hat{\mathbf{n}}\;dS = \int_s(-z\mathbf{j}-3\mathbf{k}).\left(\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{2}\right)dS$$

$$= \frac{1}{2}\int_s\left(-yz-3z\right)\,dS$$

Expressing this in spherical polar coordinates and integrating, because

$$x = 2\sin\theta\cos\phi; \qquad y = 2\sin\theta\sin\phi; \qquad z = 2\cos\theta; \qquad dS = 4\sin\theta\,d\theta\,d\phi$$

$$\therefore \int_s curl\;\mathbf{F}.\hat{\mathbf{n}}\;\;dS = \frac{1}{2}\int\int_s\left(-2\sin\theta\sin\phi\,2\cos\theta - 6\cos\theta\right)4\sin\theta\,d\theta\,d\phi$$

$$= -4\int_0^{2\pi}\int_0^{\pi/2}\left(2\sin^2\theta\cos\theta\sin\phi + 3\sin\theta\cos\theta\right)d\theta\,d\phi$$

$$= -4\int_0^{2\pi}\left[\frac{2\sin^3\theta\sin\phi}{3} + \frac{3\sin^2\theta}{2}\right]_0^{\pi/2}d\phi \qquad (2)$$

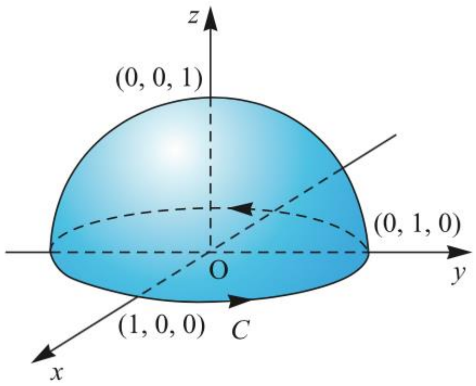
$$= -4\int_0^{2\pi}\left(\frac{2}{3}\sin\phi + \frac{3}{2}\right)\;d\phi = -12\pi$$

So we have from our two results (1) and (2)

$$\int_s curl\;\mathbf{F}.d\mathbf{S} = \oint_c \mathbf{F}.\,d\mathbf{r}$$

Example 14.2:

Verify the Stokes' Theorem for $F = (2x - y)\tilde{\boldsymbol{i}} - yz^2\tilde{\boldsymbol{j}} - y^2z\tilde{\boldsymbol{k}}$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and *C* is its boundary.



Hemispherical surface and boundary for Example 14.2

Solution:

The surface and boundary involved are illustrated in the above figure. We are required to show that

$$\oint_c \boldsymbol{F}.dr = \iint_S curl\,\boldsymbol{F}.d\boldsymbol{S}$$

Since *C* is a circle of unit radius in the *(x,y)* plane, to evaluate $\oint_c \boldsymbol{F}.dr$, we take

$x = \cos\phi,$ $\quad\quad\quad y = \sin\phi,$

so that

$r = \cos\phi\boldsymbol{i} + sin\phi\boldsymbol{j}$

giving $\quad dr = -sin\phi d\phi\, i + cos\phi d\phi\, j$

Also, on the boundary *C, z=0,* so that

$$F = (2x - y)\boldsymbol{i} = (2\,cos\phi - sin\phi)\boldsymbol{i}$$

Thus,

$$\oint_c \boldsymbol{F}.dr = \int_0^{2\pi} (2\,cos\phi - sin\phi)\boldsymbol{i}\,.\,(-sin\phi\,\boldsymbol{i} + cos\phi\,\boldsymbol{j})d\phi$$

$$= \int_0^{2\pi} (-2sin\phi\,cos\phi + sin^2\phi)d\phi = \int_0^{2\pi} \left[-sin2\phi + \frac{1}{2}(1 - cos2\phi)\right]d\phi = \pi$$

$$curl\,F = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = \boldsymbol{k}$$

The unit outward-drawn normal at a point (*x,y,z*) on the hemisphere is given by (***xi+yj+zk***) since $x^2 + y^2 + z^2 = 1$. Thus

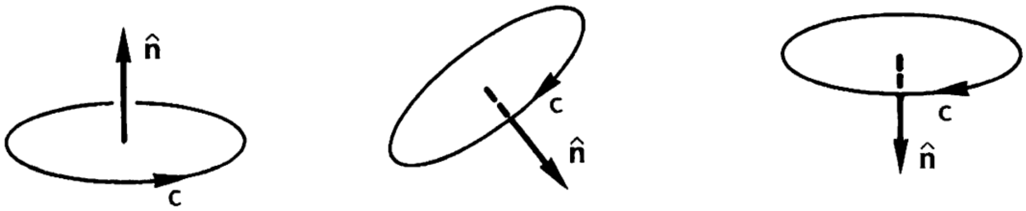
$$\iint_S curl\,\boldsymbol{F}.d\boldsymbol{S} = \iint_S \boldsymbol{k}.(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})dS$$

$$= \iint_S z\,dS = \int_0^{2\pi}\int_0^{\pi/2} cos\theta\,sin\theta\,d\theta d\phi = 2\pi\left[\frac{1}{2}sin^2\theta\right]_0^{\pi/2} = \pi$$

Hence $\oint_c \boldsymbol{F}.dr = \iint_S curl\,\boldsymbol{F}.d\boldsymbol{S}$ and Stokes' Theorem is verified.

14.2 DIRECTION OF UNIT NORMAL VECTORS TO A SURFACES

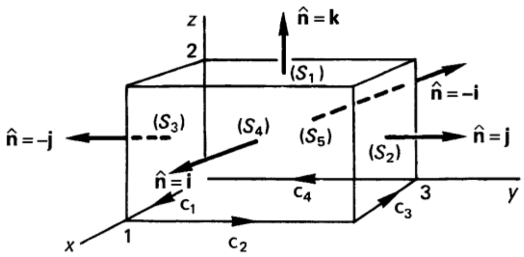
When we were dealing with the divergence theorem, the normal vectors were drawn in a direction outward from the enclosed region. With an open surface as we now have, there is in fact no inward or outward direction. With any general surface, a normal vector can be drawn in either of two opposite directions. To avoid confusion, a convention must therefore be agreed upon and the established rule is as follows.



A unit normal $\hat{\mathbf{n}}$ is drawn perpendicular to the surface *S* at any point in the direction indicated by applying the right-handed screw sense to the direction of integration round the boundary *c*. This is identical to right-hand grip rule. Having noted that point, we can now deal with the next example.

Example 14.3:

A surface consists of five sections formed by the planes x=0, x=1, y=0, y=3, z=2 in the first octant. If the vector field **F** = y**i**+z^2**j**+xy**k** exists over the surface and around its boundary, verify Stokes' theorem.



If we progress round the boundary along c_1, c_2, c_3, c_4 in an anti-clockwise manner, the normal to the surfaces will be as shown.

We have to verify that $\int_s curl\ \mathbf{F}.d\mathbf{S} = \oint_c \mathbf{F}.d\mathbf{r}$

(a) We will start off by finding $\oint_c \mathbf{F}.d\mathbf{r}$

(1) Along c_1: $y=0;\ z=0;\ dy=0;\ dz=0$

$$\therefore \int_{c1} \mathbf{F}.d\mathbf{r} = \int (0+0+0) = 0$$

(2) Along c_2: $x=1;\ z=0;\ dx=0;\ dz=0$

$$\therefore \int_{c2} \mathbf{F}.d\mathbf{r} = \int (0+0+0) = 0$$

(3) Along c_3: $y=3;\ z=0;\ dy=0;\ dz=0$

$$\therefore \int_{c3} \mathbf{F}.d\mathbf{r} = \int_1^0 (3dx+0+0) = [3x]_1^0 = -3$$

(4) Along c_4: $x=0;\ z=0;\ dx=0;\ dz=0$

$$\therefore \int_{c3} \mathbf{F}.d\mathbf{r} = \int (0+0+0) = 0$$

$$\therefore \oint_c \mathbf{F}.d\mathbf{r} = 0+0-3+0 = -3$$

$$\oint_c \mathbf{F}.d\mathbf{r} = -3$$

(b) Now we have to find $\int_s curl\ \mathbf{F}.d\mathbf{S}$

First we need an expression for curl **F.**

$$\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} + xy\mathbf{k}$$

$$curl\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z^2 & xy \end{vmatrix}$$

$$= \mathbf{i}(x-2z) - \mathbf{j}(y-0) + \mathbf{k}(0-1) = (x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}$$

Then for each section, we obtain $\int_s curl\ \mathbf{F}.d\mathbf{S} = \int curl\ \mathbf{F}.\hat{\mathbf{n}}\ dS$

(1) S_1 (top) $\hat{\mathbf{n}} = \mathbf{k}$

$$\int_{S_1} curl\ \mathbf{F}.\hat{\mathbf{n}}\,dS = \int_{S_1} \left[(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\right].(\mathbf{k})\ dS$$

$$\int_{S_1} (-1)dS = -(area\ \text{ of } S_1) = -3$$

Then, likewise

(2) S_2 (right-hand end): $\hat{\mathbf{n}} = \mathbf{j}$

$$\therefore \int_{S_2} curl\ \mathbf{F}.\hat{\mathbf{n}}\,dS = \int_{S_2} \left[(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\right].(\mathbf{j})\ dS$$

$$= \int_{S_2} (-y)\ dS$$

But y=3 for this section

$$\therefore \int_{S_2} curl\ \mathbf{F}.\hat{\mathbf{n}}\,dS = \int_{S_2} (-3)\,dS = (-3)(2) = -6$$

(3) S_3 (left-hand end): $\hat{\mathbf{n}} = -\mathbf{j}$

$$\int_{S_3} curl\ \mathbf{F}.\hat{\mathbf{n}}\,dS = \int_{S_3} \left[(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\right].(-\mathbf{j})\ dS$$

$$= \int_{S_3} (y)\ dS$$

But *y=0* over S_3

$$\therefore \int_{S_2} curl\ \mathbf{F}.\hat{\mathbf{n}}\,dS = 0$$

(4) S_4 (front): $\hat{\mathbf{n}} = \mathbf{i}$

$$\therefore \int_{S_4} curl\ \mathbf{F}.\hat{\mathbf{n}}\,dS = \int_{S_4} \left[(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\right].(\mathbf{i})\ dS$$

$$= \int_{S_4} (x-2z)\ dS$$

but *x=1* over S_4

$$\therefore \int_{S_4} curl\ \mathbf{F}.\hat{\mathbf{n}}\,dS = \int_0^3\int_0^2 (1-2z)dzdy = \int_0^3 \left[z - z^2\right]_0^2 dy$$

$$= \int_0^3 (-2)\ dy = \left[-2y\right]_0^3 = -6$$

(5) S_5 (back): $\hat{\mathbf{n}} = -\mathbf{i}$ with x=0 over S_5

Similar working to the above gives $\int_{S_5} curl\ \mathbf{F}.\hat{\mathbf{n}}\,\mathrm{dS} = 12$

Finally, collecting the five results together gives

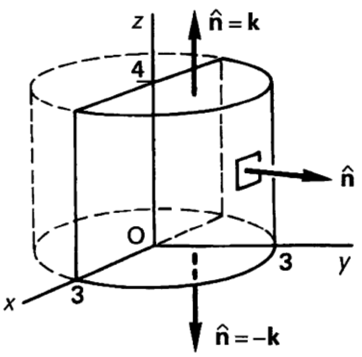
$$\therefore \int_{S_4} curl\ \mathbf{F}.\hat{\mathbf{n}}\,\mathrm{dS} = -3 - 6 + 0 - 6 + 12 = -3$$

So, referring back to our result for section (a) we see that

$$\int_s curl\ \mathbf{F}.\mathrm{d}\mathbf{S} = \oint_c \mathbf{F}.d\mathbf{r}$$

Example 14.4:

A surface S consists of that part of the cylinder $x^2 + y^2 = 9$ between z=0 and z=4 for y≥0 and the two semicircles of radius 3 in the planes z=0 and z=4. If $\mathbf{F} = z\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$, evaluate $\int_s curl\ \mathbf{F}.\mathrm{d}\mathbf{S}$ over the surface.



The surface *S* consists of three sections

(a) The curved surface of the cylinder
(b) The top and bottom semicircles

We could therefore evaluate

$$\int_s curl\ \mathbf{F}.\mathrm{d}\mathbf{S}$$

Over each of these separately.

However, we know by Stokes' theorem that

$$\mathbf{F} = z\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$$

$$\oint_c F.\ \mathrm{dr} = \oint_c (z\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}).(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz)$$

$$= \oint_c (zdx + xydy + xzdz)$$

Now we can work through this easily enough, taking c_1, c_2, c_3, c_4 in turn, and summing the results which gives

$$\int_s curl\ \mathbf{F}.\mathrm{d}\mathbf{S} = \oint_c \mathbf{F}.d\mathbf{r} = \oint_c (zdx + xydy + xzdz)$$

(1) C_1: *y=0; z=0; dy=0; dz=0*

$$\int_{C_1} \mathbf{F}.\, d\mathbf{r} = \int_{C_1} (0+0+0) = 0$$

(2) C_2: *x=-3; y=0; dx=0; dy=0*

$$\int_{C_2} \mathbf{F}.\, d\mathbf{r} = \int_{C_2} (0+0-3zdz) = \left[\frac{-3z^2}{2}\right]_0^4 = -24$$

(3) C_3: *y=-; z=4; dy=0; dz=0*

$$\int_{C_3} \mathbf{F}.\, d\mathbf{r} = \int_{C_3} (4dx+0+0) = \int_{-3}^{3} 4dx = 24$$

(4) C_4: *x=3; y=0; dx=0; dy=0*

$$\int_{C_4} \mathbf{F}.\, d\mathbf{r} = \int_{C_4} (0+0+3zdz) = \left[\frac{3z^2}{2}\right]_4^0 = -24$$

There is an alternative way of solving this example. We can consider a fictitious surface enclosed by the rectangular curve C_1-C_2-C_3-C_4 (the vertical rectangular surface formed by the closed curve). This fictitious surface shares the same closed curve as the hollow half-cylinder surface. Therefore, finding $\int_S curl\, \boldsymbol{F}.d\boldsymbol{S}$ of this fictitious surface bounded by y = 0, -3 ≤ x ≤ 3, 0 ≤ z ≤ 4, which is more straight-forward than finding the original surface integral, will also give the same answer:

$$curl\, \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & xy & xz \end{vmatrix} = (1-z)\boldsymbol{j} + y\boldsymbol{k}$$

For this vertical rectangular surface, unit normal vector, ***n*** = ***j***.

$$\int_S curl\, \boldsymbol{F}.d\boldsymbol{S} = \int_S curl\, \boldsymbol{F}.\boldsymbol{n}\, dS$$

$$= \int_{-3}^{3}\int_0^4 (1-z)dzdx$$

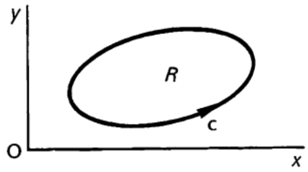
$$= \int_{-3}^{3} \left[z - \frac{z^2}{2}\right]_0^4 dx$$

$$= \int_{-3}^{3} -4\, dx = -24$$

This alternative solution demonstrates an interesting property related to Stokes' theorem: if two or more surfaces share the same closed curve, their surface integrals (of the Stokes' theorem) give the same value.

14.3 GREEN'S THEOREM (CIRCULATION FORM)

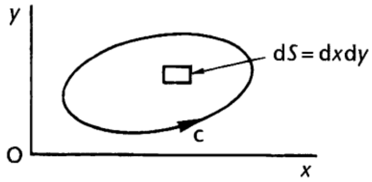
Green's theorem enables an integral over a plane area to be expressed in terms of a line integral round its boundary curve. Let *P* and *Q* be two functions of *x* and *y* that are, along with their first partial derivatives, finite and continuous inside and on the boundary *c* of a region *R* in the *x-y* plane.



If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that

$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dxdy = -\oint_c (Pdx + Qdy)$$

That is, a double integral over the plane region *R* can be transformed into a line integral over the boundary *c* of the region – and the action is reversible.



If P and Q are two single-valued functions of x and y, continuous over a plane surface S, and c is its boundary curve, then

$$\oint_c (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

where the line integral is taken round *c* in an anticlockwise manner. In vector terms, this becomes:

S is a two-dimensional space enclosed by a simple closed curve *c*.

RHS: $dS = dxdy$

$$d\mathbf{S} = \hat{\mathbf{n}}\ dS = \mathbf{k}\ dx\,dy$$

If $F = P\mathbf{i} + Q\mathbf{j}$ where *P=P(x,y)* and *Q=Q(x,y),* then

$$curl\ \mathrm{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \mathbf{i}\left(0 - \frac{\partial Q}{\partial z}\right) - \mathbf{j}\left(0 - \frac{\partial P}{\partial z}\right) + \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

But in the x-y plane, $\frac{\partial Q}{\partial z} = \frac{\partial P}{\partial z} = 0$. $\therefore$ curl $\mathbf{F} = \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

So $\int curl\ \mathbf{F}.d\mathbf{S} = \int curl\ \mathbf{F}.\hat{\mathbf{n}}dS$ and in the x-y plane, $\hat{\mathbf{n}} = \mathbf{k}$

$$\therefore \int_S curl\ \mathbf{F}.d\mathbf{S} = \int_s \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).(\mathbf{k})dS = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\therefore \int_S curl\ \mathbf{F}.d\mathbf{S} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \qquad (1)$$

Now by Stokes' theorem

$$\int_S curl\ \mathbf{F}.d\mathbf{S} = \oint_c \mathbf{F}.d\mathbf{r}$$

LHS: in this case $\oint_c \mathbf{F}.d\mathbf{r} = \oint_c (P\mathbf{i} + Q\mathbf{j}).(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz)$

$$= \oint_c (Pdx + Qdy)$$

$$\therefore \oint_c \mathbf{F}.d\mathbf{r} = \oint_c Pdx + Qdy \qquad (2)$$

Therefore from (1) and (2)

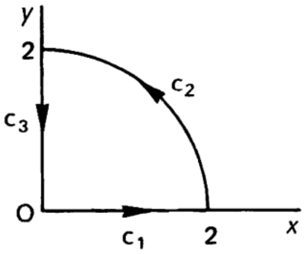
Stokes' theorem $\int_S curl\ \mathbf{F}.d\mathbf{S} = \oint_c \mathbf{F}.\ d\mathbf{r}$ in two dimensions becomes Green's theorem

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_c Pdx + Qdy$$

Example 14.5:

Verify Green's theorem for the integral $\oint_c \left[(x^2 + y^2) dx + (x + 2y) dy \right]$ taken round the boundary curve c defined by

$y = 0$	$0 \le x \le 2$
$x^2 + y^2 = 4$	$0 \le x \le 2$
$x = 0$	$0 \le y \le 2$



Green's theorem: $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \oint_c (\mathrm{P\ dx + Q\ dy})$

In this case $(x^2 + y^2) dx + (x + 2y) dy = P\,\mathrm{dx} + \mathrm{Q\,dy}$

$\therefore P = x^2 + y^2$ and $Q = x + 2y$

We now take c_1, c_2, c_3 in turn.

(1) c_1: *y=0; dy=0*

$$\therefore \int_{c_1} (P\ \mathrm{dx} + \mathrm{Q\,dy}) = \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

(2) c_2: $x^2 + y^2 = 4 \quad \therefore y^2 = 4 - x^2 \quad \therefore \mathrm{y} = (4\text{ - x}^2)^{1/2}$

$$x + 2y = x + 2(4 - x^2)^{1/2}$$

$$dy = \frac{1}{2}(4 - x^2)^{-1/2}(-2x)dx = \tfrac{-x}{\sqrt{4-x^2}}dx$$

$$\therefore \int_{c_2} (P\ \mathrm{dx}\ \ + \mathrm{Q\,dy}) = \int_{c_2} \left[4 + \left(x + 2\sqrt{4 - x^2}\right)\left(\frac{-x}{\sqrt{4 - x^2}}\right)\right] dx$$

$$= \int_{c_2} \left[4 - 2x - \frac{x^2}{\sqrt{4 - x^2}}\right] dx$$

Putting x = 2 sin θ $\sqrt{4 - x^2} = 2\cos\theta$ $\qquad dx = 2\cos\theta d\theta$

Limits: x=2 $\theta = \frac{\pi}{2}$; x=0 , θ=0

$$\therefore \int_{c_2} (P\,\mathrm{dx} + \mathrm{Q\,dy}) = \int_{\pi/2}^{0} \left[4 - 4\sin\theta - \frac{4\sin^2\theta}{2\cos\theta}\right] 2\cos\theta d\theta$$

$$= 4\left[2\sin\theta - \sin^2\theta - \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{\pi/2}^{0}$$

$$= 4\left[-\left(2 - 1 - \frac{\pi}{4}\right)\right] = \pi - 4$$

Finally

(3) c_3: x=0; dx=0

$$\therefore \int_{c_3} (\mathrm{P\,dx} + \mathrm{Q\,dy}) = \int_2^0 2y\ \ \mathrm{dy} = \left[\mathrm{y}^2\right]_2^0 = -4$$

Therefore, collecting our three partial results

$$\oint_c (P\ \mathrm{dx} + \mathrm{Q\,dy}) = \frac{8}{3} + \pi - 4 - 4 = \pi - \frac{16}{3}$$

That is one part done. Now we have to evaluate $\iint_s \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$P = x^2 + y^2 \quad \therefore \frac{\partial \mathrm{P}}{\partial \mathrm{y}} = 2y$$

$$Q = x + 2y \quad \therefore \frac{\partial Q}{\partial x} = 1$$

$$\iint_s \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \iint_S (1 - 2y)\, dy\, dx$$

It will be more convenient to work in polar coordinates, so we make the substitutions

$$x = r\cos\theta; \qquad y = r\sin\theta; \qquad dS = dxdy = r\, dr\, d\theta$$

$$\therefore \iint_s \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_0^{\pi/2} \int_0^2 (1 - 2r\sin\theta)\ \mathrm{r\ dr\, d}\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{2r^3}{3}\sin\theta \right]_0^2 d\theta$$

$$= \int_0^{\pi/2} \left[2 - \frac{16}{3}\sin\theta \right] d\theta = \left[2\theta + \frac{16}{3}\cos\theta \right]_0^{\pi/2} = \pi - \frac{16}{3}$$

So we have established once again that

$$\oint_c (P\,\mathrm{dx} + \mathrm{Q\,dy}) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$