VECTOR ALGEBRA I

WEEK 3: VECTOR ALGEBRA I

3.1 INTRODUCTION

In the world of engineering, physical quantities can be divided mainly into scalar and vector. These quantities can be represented by numbers alone (i.e., magnitude only), with the appropriate units, and they are called **scalars.** Another physical quantity with magnitude and direction are called **vectors**. Scalars and vectors are the underlying elements in vector analysis.

Scalar vs Vector

	Scalar	Vector
Example	Mass; length; temperature; voltage	Displacement; velocity; force; acceleration
Unit of quantities	kg; m; Degree; Volt	m; ms^{-1}; N; ms^{-1}
Direction	No	Yes
Symbol/Notation	a; b; A; B; PQ	$\underset{\sim}{a}$; $\underset{\sim}{b}$; $\overrightarrow{OA}$, $\overrightarrow{\mathrm{OB}}$, $\overrightarrow{\mathrm{PQ}}$

3.2 BASIC CONCEPTS

A **scalar** is a quantity that is determined by its magnitude. It takes on a numerical value, i.e., a number. Examples of scalars are time, temperature, length, distance, speed, density, energy, and voltage.

A **vector** is a quantity that has both magnitude and direction. We can say that a vector is an *arrow* or a *directed line segment*. For example, a velocity vector has length or magnitude, which is speed, and direction, which indicates the direction of motion (**Fig 3.1**); a force vector points in the direction in which the force acts and its length is a measure of the force's strength.

A vector (arrow) has a tail, called its initial point, and a tip, called its terminal point. The length of the arrow equals the distance between initial point and terminal point (**Fig 3.1**). This is called the length (or *magnitude*) of the vector a and is denoted by $|a|$. Another name for *length* is norm (or *Euclidean norm*). A vector of length 1 is called a unit vector.

DEFINITIONS The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

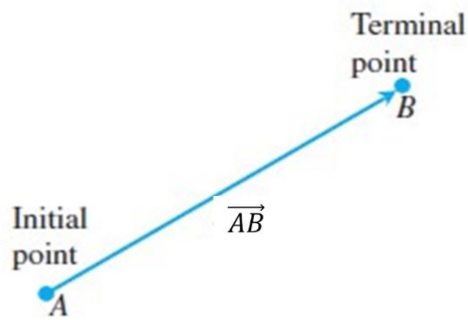


Fig 3.1: The directed line segment $\overrightarrow{AB}$ is called a vector

DEFINITION

If **v** is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If **v** is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

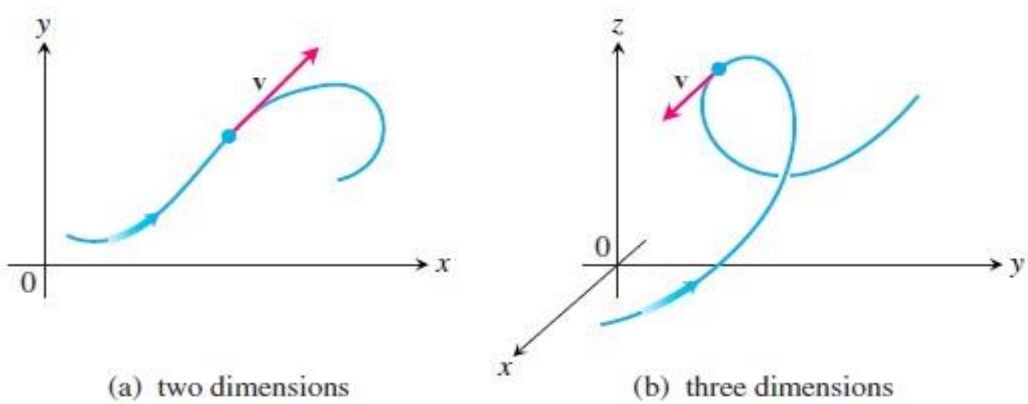


Fig 3.2: The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

Equality of Vectors - Two vectors a and b are equal, written, if they have the same length and the same direction as shown in **Fig. 3.3**.

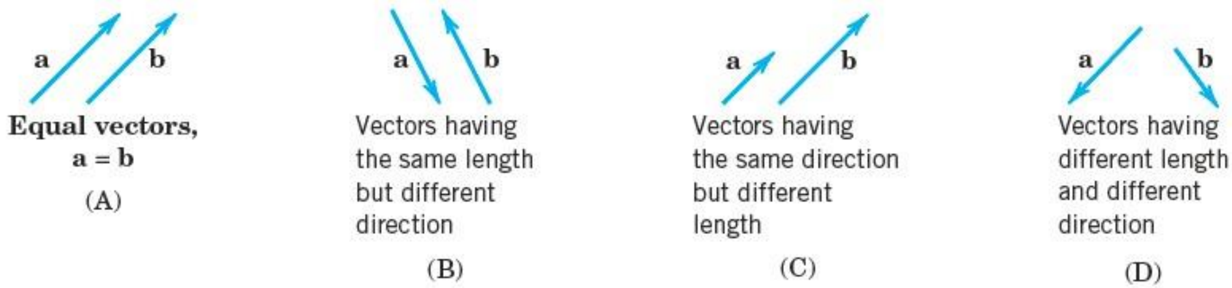


Fig. 3.3 (A) Equal Vectors. (B) – (D) Different Vectors

3.2.1 COMPONENTS OF A VECTOR

Let ***a*** be a given vector with initial point *P*: (x_1, y_1, z_1) and terminal point *Q*: (x_2, y_2, z_2). Then the three coordinate differences

$$a_1 = x_2 - x_1, \qquad a_2 = y_2 - y_1, \qquad a_3 = z_2 - z_1$$

are called the **components** of the vector **a** with respect to that coordinate system, and we write simply

a = $[a_1, a_2, a_3]$. See **Fig 3.4 (a).** The **length** $|a|$ of **a** can now readily be expressed in terms of components and the Pythagorean Theorem we have

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

A Cartesian coordinate system being given, the position vector r of a point *A:* (*x*, *y*, *z*) is the vector with the origin (0, 0, 0) as the initial point and *A* as the terminal point (See **Fig 3.4 (b)**).

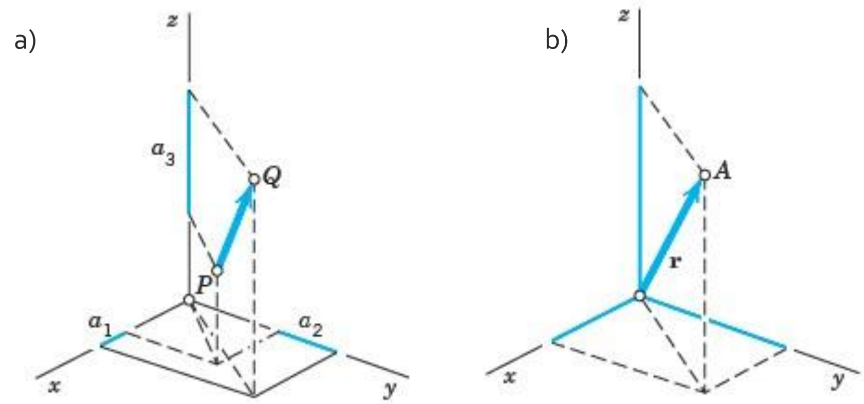


Fig 3.4 (a) Components of a vector **(b)** Position vector **r** of a point *A*: (*x*, *y*, *z*)

Example 3.1

Components and Length of a Vector

The vector **a** with initial point *P*: (4, 0, 2) and terminal point *Q*: (6, −1, 2) has the components

$$a_1 = 6 - 4 = 2, \qquad a_2 = -1 - 0 = -1, \qquad a_3 = 2 - 2 = 0.$$

Hence **a** = $\langle 2, -1, 0 \rangle$ Equation gives the length

$$|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

If we choose (−1, 5, 8) as the initial point of **a**, the corresponding terminal point is (1, 4, 8).

If we choose the origin (0, 0, 0) as the initial point of **a**, the corresponding terminal point is (2, −1, 0); its coordinates equal the components of **a**. This suggests that we can determine each point in space by a vector, called the *position vector* of the point, as follows.

Exercises

Let u= $3i - 2j$ and $= -2i + 5j$. Find the **(a)** component form and **(b)** magnitude (length) of the vector.

1. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$ **2.** $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$

3.2.2 VECTOR ADDITION, SCALAR MULTIPLICATION

Two principal operations involving vectors are ***vector addition*** and ***scalar multiplication***. A scalar is simply a real number, and is called such when we want to draw attention to its differences from vectors. Scalars can be positive, negative, or zero and are used to "scale" a vector by multiplication.

Addition of Vectors

The **sum** $\mathbf{a} + \mathbf{b}$ of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is obtained by adding the corresponding components,

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, \quad a_2 + b_2, \quad a_3 + b_3].$$

Geometrically, place the vectors as in **Fig. 3.5** (the initial point of **b** at the terminal point of **a**); then **a + b** is the vector drawn from the initial point of **a** to the terminal point of **b**. **Fig. 3.5** also shows (for the plane) that the "algebraic" way and the "geometric way" of vector addition give the same vector.

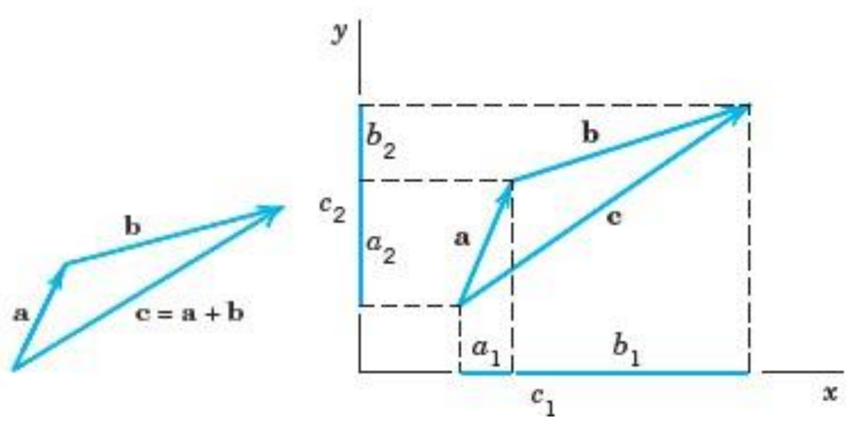
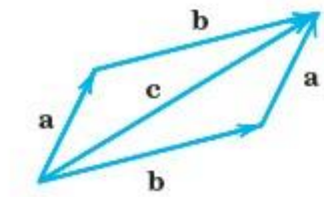


Fig 3.5 Vector Additions

Basic Properties of Vector Addition

(a)	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	(*Commutativity*)
(b)	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	(*Associativity*)
(c)	$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$	
(d)	$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$	

Properties (a) and (b) are verified geometrically in **Fig. 3.6** and **Fig 3.7**, respectively. Furthermore, **-a denotes** the vector having the length $|\boldsymbol{a}|$ and the direction opposite to that of **a**.



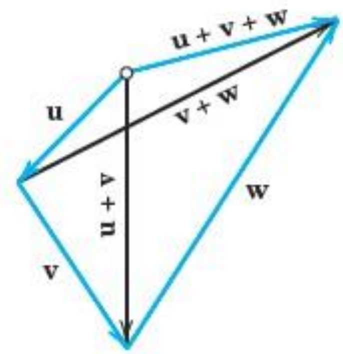


Fig 3.6 Commutativity of vector addition **Fig 3.7 A**ssociativity of vector addition

Scalar Multiplication (Multiplication by a Number)

The product $c\mathbf{a}$ of any vector $\mathbf{a} = [a_1, a_2, a_3]$ and any scalar c (real number c) is the vector obtained by multiplying each component of **a** by c,

$$c\mathbf{a} = [ca_1, ca_2, ca_3].$$

Geometrically, if **a ≠ 0** then c**a** with $c > 0$ has the direction of **a** and with $c < 0$ the direction opposite to **a**. In any case, the length of c**a** is $|c\boldsymbol{a}| = |c|\ |\boldsymbol{a}|$, and c**a = 0** if **a = 0** or $c = 0$ (or both) (See **Fig 3.8**).

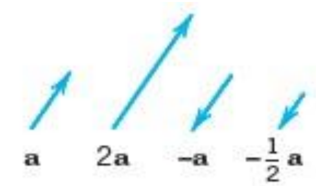


Fig 3.8 Scalar multiplication [multiplication of vectors by scalars (numbers)]

Basic Properties of Scalar Multiplication

From the definitions we obtain directly:

$$\begin{aligned}
&\text{(a)} \quad c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b} \\
&\text{(b)} \quad (c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a} \\
&\text{(c)} \quad c(k\mathbf{a}) = (ck)\mathbf{a} \qquad \text{(written } ck\mathbf{a}\text{)} \\
&\text{(d)} \quad 1\mathbf{a} = \mathbf{a}.
\end{aligned}$$

Example 3.2

With respect to a given coordinate system, let

$$\mathbf{a} = [4, 0, 1] \quad \text{and} \quad \mathbf{b} = [2, -5, \tfrac{1}{3}].$$

Then $-\mathbf{a} = [-4, 0, -1]$, $7\mathbf{a} = [28, 0, 7]$, $\mathbf{a} + \mathbf{b} = [6, -5, \tfrac{4}{3}]$, and

$$2(\mathbf{a} - \mathbf{b}) = 2[2, 5, \tfrac{2}{3}] = [4, 10, \tfrac{4}{3}] = 2\mathbf{a} - 2\mathbf{b}.$$

Exercises

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

3.2.3 UNIT VECTOR

A vector ***v*** of length 1 is called a **unit vector**. In this representation, **i**, **j**, **k** are the unit vectors in the positive directions of the axes of a Cartesian coordinate system. The **standard unit vectors** are

$$i = \langle 1, 0, 0 \rangle, \quad j = \langle 0,1,0 \rangle \quad k = \langle 0, 0,1 \rangle$$

Any vector can be written as a *linear combination* of the standard unit vectors as follows:

$$v = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle$$

$$= v_1\langle 1,0,0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0,1 \rangle$$

$$v = v_1 i + v_2 j + v_3 k$$

From **Figure 3.9**, we call the scalar (or number) v_1 the **i-component** of the vector ***v***, v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

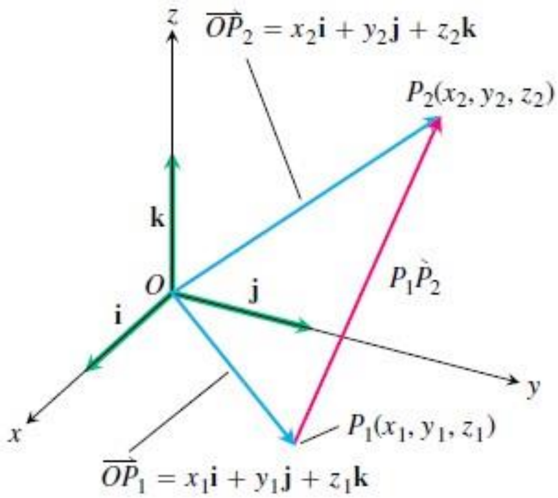


Fig. 3.9 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2}$

Whenever v$\neq 0$, its length $|v|$ is not zero and

$$\left|\frac{1}{|v|}v\right| = \frac{1}{|v|}|v| = 1$$

That is, $v/|v|$ is a unit vector in the direction of ***v***, called **the direction** of the nonzero vector ***v***.

Example 3.3

If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times a unit vector in the direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ has the same direction as $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underbrace{5}_{\text{Length (speed)}}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$

In summary, we can express any nonzero vector ***v*** in terms of its two important features, length and direction, by writing

$$\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$$

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

<u>Example 3.4</u>

Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

The unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$.

***PROVE that the length of unit vector is 1.

3.3 VECTOR IN SPACE

3.3.1 Cartesian coordinates of a vector in 2D space & its polar expressions

<u>(a) Definition of a 2D vector</u>

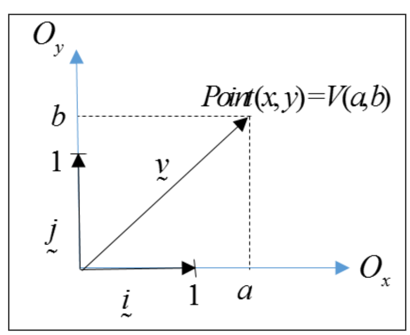
Let O be the origin and let O_x and O_y be two mutually perpendicular coordinate axes.

Then, the plane containing O_x and O_y is called the xy-plane or the xy-coordinate system and O_x is called the x axis and O_y is y axis.

The vector $\underset{\sim}{i}$ is the vector from the origin O to the point $(1,0)$.

The vector $\underset{\sim}{j}$ is the vector from the origin O to the point $(0,1)$.

Note: $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors and also position vectors.



Any vector $\underset{\sim}{v}$ in xy-plane can be represented by $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j}$ or $\underset{\sim}{v} = \langle a, b \rangle$ where a and b are scalars. The scalars a and b are called the components of the vector $\underset{\sim}{v}$ with respect to that coordinate system.

The vector $a\underset{\sim}{i}$ and vector $b\underset{\sim}{j}$ are called the vector components in the direction of $\underset{\sim}{i}$ and $\underset{\sim}{j}$, respectively.

Notation:

(i) The vector $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j}$ can be denoted by $\underset{\sim}{v} = \langle a, b \rangle$

(ii) The point P at (a, b) can be denoted by (a, b), $P\ (a, b)$ or $P = (a, b)$

(iii) Note that $(a, b) \neq \langle a, b \rangle$ to avoid confusion. (a, b) represent coordinates of a point. $\langle a, b \rangle$ represent components of a vector.

(b) Vector Algebra of a 2D vector

Let $\underset{\sim}{v}_1 = a_1\underset{\sim}{i} + b_1\underset{\sim}{j}$ and $\underset{\sim}{v}_2 = a_2\underset{\sim}{i} + b_2\underset{\sim}{j}$ be two vectors. Then

(i) $\underset{\sim}{v}_1 = \underset{\sim}{v}_2 >>$ Then, $a_1 = a_2; b_1 = b_2$

(ii) $\underset{\sim}{v}_1 + \underset{\sim}{v}_2 >>$ Then, $(a_1 + a_2)\underset{\sim}{i} + (b_1 + b_2)\underset{\sim}{j}$

(iii) $\underset{\sim}{v}_1 - \underset{\sim}{v}_2 >>$ Then, $(a_1 - a_2)\underset{\sim}{i} + (b_1 - b_2)\underset{\sim}{j}$

(iv) Let α is a scalar, then $\alpha\underset{\sim}{v}_1 = (\alpha a_1)\underset{\sim}{i} + (\alpha b_1)\underset{\sim}{j}$

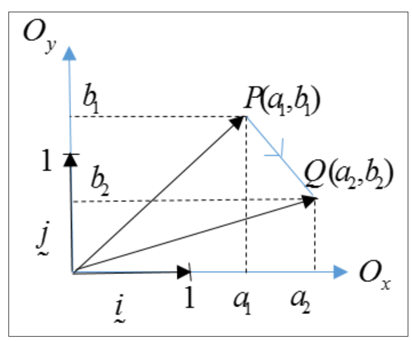
(c) Theorem of an arbitrary vector in 2D space

Let P and Q be the points (a_1, b_1) and (a_2, b_2) respectively. Then, the vector $\overrightarrow{PQ}$ is given by
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle (a_2 - a_1), (b_2 - b_1) \rangle$

Proof:

Let $\overrightarrow{OP} = \langle a_1, b_1 \rangle$, $\overrightarrow{OQ} = \langle a_2, b_2 \rangle$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle (a_2 - a_1), (b_2 - b_1) \rangle$



(d) Magnitude & Angle of a vector in 2D space

Let $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j}$ be a 2D vector.

(i) The magnitude of $\underset{\sim}{v}$ is defined as $|\underset{\sim}{v}| = \sqrt{a^2 + b^2}$

(ii) The angle between $\underset{\sim}{v}$ and a line parallel to the x-axis is defined as $\theta = tan^{-1}\frac{b}{a}$

Hint: Identify the quadrant; θ is positive if it is measured in the direction of anti-clockwise; θ is negative if it is measured in the direction of clockwise.

(e) Transformation of Cartesian form of a 2D vector to polar form

By using magnitude and angle of a vector, the Cartesian form of a vector (i.e., $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j}$) can be transformed into polar form (i.e., $\underset{\sim}{v} = \underbrace{|\underset{\sim}{v}|}_{magnitude} (cos(\underbrace{\theta}_{angle})\underset{\sim}{i} + sin(\underbrace{\theta}_{angle})\underset{\sim}{j})$

Thus, we have $\underset{\sim}{v} = \underbrace{a\underset{\sim}{i} + b\underset{\sim}{j}}_{Cartesian\ domain} = \underbrace{|\underset{\sim}{v}|(cos(\theta)\underset{\sim}{i} + sin(\theta)\underset{\sim}{j})}_{Polar\ domain}$

Exercise

(i) Let $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ be position vectors of the points $U\ (2,3)$, $V\ (1,5)$ and $W\ (3,-4)$, respectively. Find

(a) $\underset{\sim}{z} = \underset{\sim}{u} - 2\underset{\sim}{v} + 3\underset{\sim}{w}$

(b) the magnitude of $\underset{\sim}{z}$

(c) the angle between $\underset{\sim}{z}$ and O_x

(d) transform the vector $\underset{\sim}{z}$ from Cartesian domain into Polar domain

(e) compare the result in (a) and (d), explain your finding and relate this in the application of engineering.

(ii) Determine the unit vector in the direction of $\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

(iii) Find the unit vector from the point $P\ (1,4)$ to the point $Q\ (3,-5)$

(iv) Find a vector of magnitude 3 in the direction of $\underset{\sim}{v} = -\underset{\sim}{i} + 3\underset{\sim}{j}$

3.3.2 Cartesian coordinates of a vector in 3D space (Volume) & its polar expression

(a) Definition of a 3D vector

Let O be the origin and let O_x , O_y and O_z be three mutually perpendicular coordinate axes.

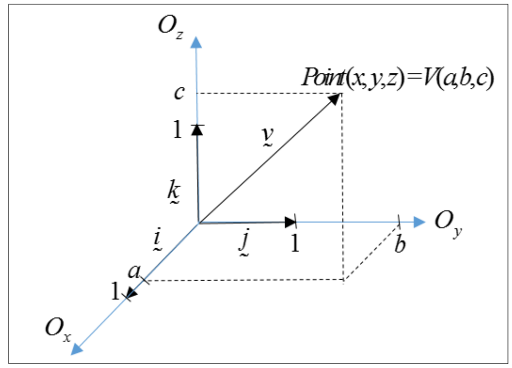
Then, the plane containing O_x , O_y and O_z is called the xyz-plane or the xyz-coordinate system (*Follow right hand rule*) and O_x is called the x axis, O_y is y axis and O_z is z axis.

The vector $\underset{\sim}{i}$ is the vector from the origin O to the point $(1,0,0)$.

The vector $\underset{\sim}{j}$ is the vector from the origin O to the point $(0,1,0)$.

The vector $\underset{\sim}{k}$ is the vector from the origin O to the point $(0,0,1)$.

Note: $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors and also position vectors.



Any vector $\underset{\sim}{v}$ in xyz-plane can be represented by $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ or $\underset{\sim}{v} = \langle a, b, c\rangle$ where a, b and c are scalars. The scalars a, b and c are called the components of the vector $\underset{\sim}{v}$ with respect to that coordinate system.

The vector $a\underset{\sim}{i}$, vector $b\underset{\sim}{j}$ and vector $c\underset{\sim}{k}$ are called the vector components in the direction of $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ respectively.

Notation:

(i) The vector $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ can be denoted by $\underset{\sim}{v} = \langle a, b, c\rangle$

(ii) The point P at (a, b, c) can be denoted by (a, b, c), $P(a, b, c)$ or $P = (a, b, c)$

(iii) Note that $(a, b, c) \neq \langle a, b, c\rangle$ to avoid confusion. (a, b, c) represent coordinates of a point. $\langle a, b, c\rangle$ represent components of a vector.

(b) Vector Algebra of a 3D vector

Let $\underset{\sim}{v}_1 = a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k}$ and $\underset{\sim}{v}_2 = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$ be two vectors. Then

(i) $\underset{\sim}{v}_1 = \underset{\sim}{v}_2$ >> Then, $a_1 = a_2$; $b_1 = b_2$; $c_1 = c_2$

(ii) $\underset{\sim}{v}_1 + \underset{\sim}{v}_2$ >> Then, $(a_1 + a_2)\underset{\sim}{i} + (b_1 + b_2)\underset{\sim}{j} + (c_1 + c_2)\underset{\sim}{k}$

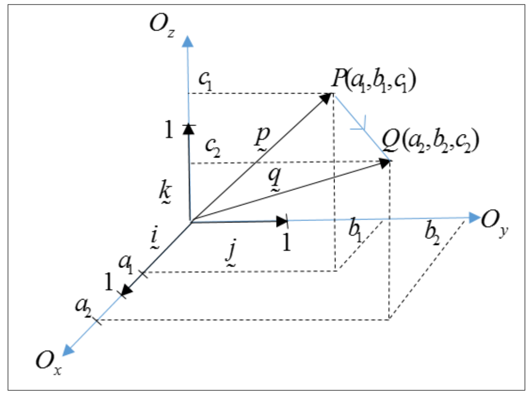
(iii) $\underset{\sim}{v}_1 - \underset{\sim}{v}_2$ >> Then, $(a_1 - a_2)\underset{\sim}{i} + (b_1 - b_2)\underset{\sim}{j} + (c_1 - c_2)\underset{\sim}{k}$

(iv) Let α is a scalar, then $\alpha\underset{\sim}{v}_1 = (\alpha a_1)\underset{\sim}{i} + (\alpha b_1)\underset{\sim}{j} + (\alpha c_1)\underset{\sim}{k}$

(c) Theorem of an arbitrary vector in 3D space

Let P and Q be the points (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively. Then, the vector $\overrightarrow{PQ}$ is given by

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle (a_2 - a_1), (b_2 - b_1), (c_2 - c_1) \rangle$$



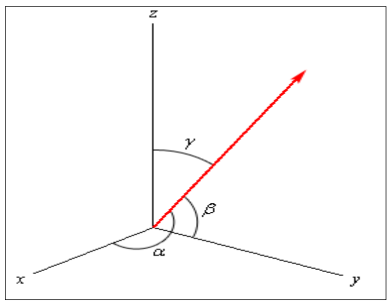
Proof:

Let $\overrightarrow{OP} = \langle a_1, b_1, c_1 \rangle$, $\overrightarrow{OQ} = \langle a_2, b_2, c_2 \rangle$,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle (a_2 - a_1), (b_2 - b_1), (c_2 - c_1) \rangle$$

(d) Magnitude & Angle of a vector in 3D space

Let $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ be a 3D vector and let α, β, and γ be the direction angles of $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$



The magnitude and angle that define vector $\underset{\sim}{v}$ can be obtained as following:

(i) The magnitude of $\underset{\sim}{v}$ is defined as $|\underset{\sim}{v}| = \sqrt{a^2 + b^2 + c^2}$

(ii) The angle between $\underset{\sim}{v}$ and a line parallel to the x-axis is defined as $\alpha = cos^{-1}\frac{a}{|\underset{\sim}{v}|}$;

The angle between $\underset{\sim}{v}$ and a line parallel to the y-axis is defined as $\beta = cos^{-1}\frac{b}{|\underset{\sim}{v}|}$;

The angle between $\underset{\sim}{v}$ and a line parallel to the z-axis is defined as $\gamma = cos^{-1}\frac{c}{|\underset{\sim}{v}|}$.

(e) Transformation of Cartesian form of a 3D vector to polar form

By using magnitude and angle of a vector, the Cartesian form of a vector $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ can be transformed into polar form $\underset{\sim}{v} = |\underset{\sim}{v}|(cos\,\alpha\,\underset{\sim}{i} + cos\,\beta\,\underset{\sim}{j} + cos\,\gamma\,\underset{\sim}{k})$.

Thus, we have $v = \underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k} = |v|\left(cos\,\alpha\,\underset{\sim}{i} + cos\,\beta\,\underset{\sim}{j} + cos\,\gamma\,\underset{\sim}{k}\right)$

(f) Important remarks for polar form of a 3D vector

(i) The unit vector $\hat{\underset{\sim}{v}}$ is $\frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = (cos\,\alpha\,\underset{\sim}{i} + cos\,\beta\,\underset{\sim}{j} + cos\,\gamma\,\underset{\sim}{k})$ or $\langle cos\,\alpha, cos\,\beta, cos\,\gamma \rangle$

(ii) Magnitude of a unit vector, $\hat{\underset{\sim}{v}}$ is 1. Thus, we get $cos^2\,\alpha + cos^2\,\beta + cos^2\,\gamma = 1$

(iii) The direction angles of negative vector, -$\underset{\sim}{v}$ are $\pi - \alpha, \pi - \beta, \pi - \gamma$

(iv) Have a clear definition for the following term:

Direction angles	Direction cosines	Direction ratio
α ,β, and γ are called the *direction angles* of $\underset{\sim}{v}$	$\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of $\underset{\sim}{v}$	The ratios a: b: c is called the *direction ratio* of $\underset{\sim}{v}$
For polar coordinate $\underset{\sim}{v} = \lvert\underset{\sim}{v}\rvert(\cos\alpha\,\underset{\sim}{i} + \cos\beta\,\underset{\sim}{j} + \cos\gamma\,\underset{\sim}{k})$		For Cartesian coordinate $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$

Additional remarks:

(i) If $\cos^2\alpha + \cos^2\beta + \cos^2\gamma \neq 1$, then there does exist a <u>*unit vector*</u> with the direction cosines $\langle \cos\alpha + \cos\beta + \cos\gamma \rangle$.

Note: because unit vector has magnitude of 1.

(ii) Two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ have the same <u>*direction cosines*</u> if and only if they have the same direction.

Note: Different direction cosines shows different directions.

(iii) Two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ have the same <u>*direction ratios*</u> if and only if they are parallel (i.e. $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are in the same direction or in opposite directions).

Note: As explained by the scalar multiplication and parallel vector.

<u>Exercise</u>

Let u, v and w be position vectors of the points U $(2,3,1)$, V $(0,-5,1)$ and $W(-3,0,0)$, respectively. Find

(i) $\underset{\sim}{z} = \underset{\sim}{u} - 2\underset{\sim}{v} + 3\underset{\sim}{w}$

(ii) transform $\underset{\sim}{z}$ from Cartesian domain (i.e., $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$) to Polar domain (i.e.,$r$ $(\cos\alpha\,\underset{\sim}{i} + \cos\beta\,\underset{\sim}{j} + \cos\gamma\,\underset{\sim}{k})$ where r is its magnitude.

(iii) the angle between $\underset{\sim}{z}$ and O_x

(iv) direction cosines of $\underset{\sim}{z}$ in three directions $\underset{\sim}{i}$,$\underset{\sim}{j}$ and $\underset{\sim}{k}$.

(v) unit vector of $\underset{\sim}{z}$

(vi) If given direction angle as following, can you identify whether the vector with the following direction cosine $(\cos\alpha\,\underset{\sim}{i} + \cos\beta\,\underset{\sim}{j} + \cos\gamma\,\underset{\sim}{k})$ is exist or not?

------ vector $\underset{\sim}{m}$ has direction angle α,β, and γ of $(\pi/4, 2\pi/3, \pi/3)$.

------ vector $\underset{\sim}{n}$ has direction angle α,β, and γ of $(\pi/2, \pi/3, \pi/3)$.

(vii) Find the direction cosines of negative vector $-\underset{\sim}{z}$. Then find the relationship between the direction cosines of vector $\underset{\sim}{z}$ and $-\underset{\sim}{z}$.

3.4 GRADIENT, DIVERGENCE, CURL OF VECTOR FIELD

3.4.1 GRADIENT OF VECTOR FIELD

Using scalar fields instead of vector fields is of a considerable advantage because scalar fields are easier to use than vector fields. It is the "gradient" that allows us to obtain vector fields from scalar fields, and thus the gradient is of great practical importance to the engineer. Gradients are useful in several ways, notably in giving the rate of change of in any direction in space, in obtaining surface normal vectors, and in deriving vector fields from scalar fields.

Gradient

The setting is that we are given a scalar function $f(x, y, z)$ that is defined and differentiable in a domain in 3-space with Cartesian coordinates x, y, z. We denote the **gradient** of that function by grad f or ∇f (read **nabla** f). Then the qradient of $f(x, y, z)$ is defined as the vector function

$$grad\ f = \nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

The notation ∇f is suggested by the *differential operator* ∇ (read *nabla*) defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Example 3.5

If $f(x, y, z) = 2y^3 + 4xz + 3x$, then $\boldsymbol{grad\ f} = [4z + 3, 6y^2, 4x]$

Exercises

Find the gradient of the following function at the given point.

(a) $f(x, y) = \ln(x^2 + y^2)$ at point (1, 1)

(b) $f(x, y) = \sqrt{2x + 3y}$ at point (-1, 2)

3.4.2 DIRECTIONAL DERIVATIVES

From gradient we know that the partial derivatives give the rates of change of *f(x, y, z)* in the directions of the three coordinate axes. It seems natural to extend this and ask for the rate of change of *in an arbitrary direction* in space. This leads to the concept of directional derivative.

Directional Derivative

The directional derivative $D_{\mathbf{b}}f$ or df/ds of a function $f(x, y, z)$ at a point P in the direction of a vector $\mathbf{b}$ is defined by Figure 3.10

$$D_{\mathbf{b}}f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}.$$

Here Q is a variable point on the straight line L in the direction of $\mathbf{b}$, and $|s|$ is the distance between P and Q. Also, $s > 0$ if Q lies in the direction of $\mathbf{b}$ (as in (Fig.) $s < 0$ if Q lies in the direction of $-\mathbf{b}$, and $s = 0$ if $Q = P$.

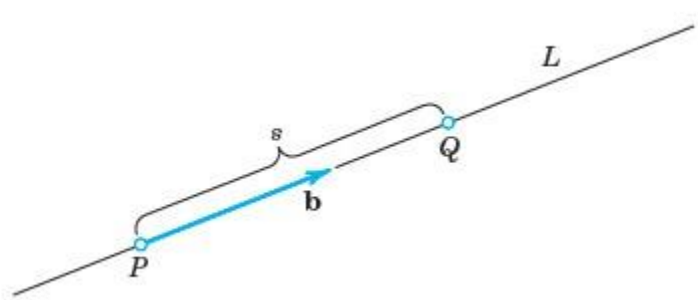


Fig. 3.10 Directional Derivative (Refer to above Equation)

The above equation can be derived into

$$\left(\frac{df}{ds}\right)_{b,P} = \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_P \boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)_P \boldsymbol{j}\right]}_{\text{Gradient of } f \text{ at } P} \cdot \underbrace{[b_1\boldsymbol{i} + b_2\boldsymbol{j}]}_{\text{Direction b}}$$

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad *f*" as well as "gradient of *f*" and "del *f*." The symbol ∇ by itself is read "del." Another notation for the gradient is grad *f*.

The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P\ (x\ , y\ )$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{b},\,P} = (\nabla f)_P \cdot \mathbf{b}$$

the dot product of the gradient ∇f at P and $\mathbf{b}$

Example 3.6

Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at P: (2, 1, 3) in the direction of $\mathbf{a} = [1, 0, -2]$.

Solution. grad $f = [4x, 6y, 2z]$ gives at P the vector grad $f(P) = [8, 6, 6]$. From this we obtain, since $|\mathbf{a}| = \sqrt{5}$,

$$D_{\mathbf{a}}f(P) = \frac{1}{\sqrt{5}}[1, 0, -2] \bullet [8, 6, 6] = \frac{1}{\sqrt{5}}(8 + 0 - 12) = -\frac{4}{\sqrt{5}} = -1.789.$$

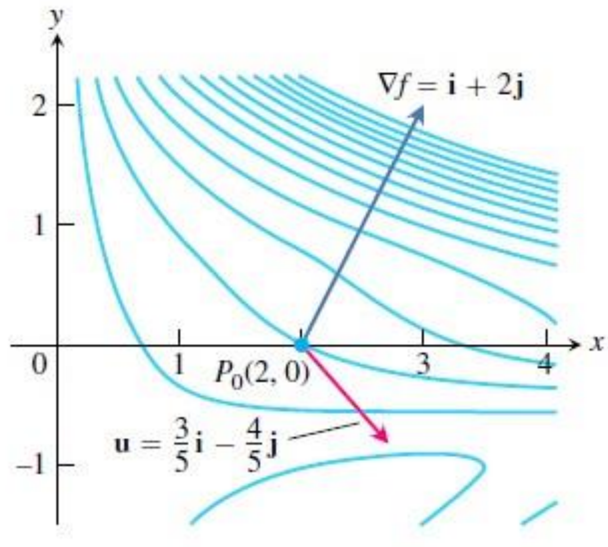
The minus sign indicates that at P the function f is decreasing in the direction of $\mathbf{a}$.

Example 3.7

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$



Picture ∇f as a vector in the domain of f. The figure shows a number of level curves of f. The rate at which f changes at (2, 0) in the direction $\mathbf{u} = (3/5)\mathbf{i} - (4/5)\mathbf{j}$ is $\nabla f \cdot \mathbf{u} = -1$

Exercises

Find the derivative of the function at P_o in the direction of u _

(a) $g(x, y) = \frac{x-y}{xy+2}$, $P_o(1, -1)$, $u = 12i + 5j$

(b) $h(x, y, z) = \cos xy + e^{yz} + \ln(zx)$, $P_o(1, 0, 1/2)$, $u = 1i + 2j + 2k$

(c) $h(x, y, z) = 3e^{x}\cos yz$, $P_o(0, 0, 0)$, $u = 2i + j - 2k$

3.4.3 DIVERGENCE OF VECTOR FIELD

From a scalar field we can obtain a vector field by the gradient. Conversely, from a vector field we can obtain a scalar field by the divergence or another vector field by the curl.

To begin, let $v(x, y, z)$ be a differentiable vector function, where *x*, *y*, *z* are Cartesian coordinates, and let v_1, v_2, v_3 be the components of **v**. Then the function

$$div\ v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the **divergence** of **v** or the *divergence of the vector field defined by* **v**. For example, if

$$v = [3xz, 2xy, -yz^2] = 3xzi + 2xyj - yz^2k$$

Then

$$div\, v = 3z + 2x + 2yz$$

Another common notation for the divergence is

$$\text{div v} = \nabla \bullet v = \left[\frac{\partial}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial x}\right] \bullet [v_1, v_2, v_3]$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \bullet [v_1 i + v_2 j + v_3 k]$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

With understanding that the "product" $\left(\frac{\partial}{\partial x}\right) v_1$ in the dot product means the partial derivative $\frac{\partial v_1}{\partial x}$, etc. This is a convenient notation, but nothing more. Note that $\boldsymbol{\nabla . v}$ means the scalar div **v**, whereas $\boldsymbol{\nabla f}$ means the vector grad *f*.

<u>Example 3.8</u>

$$If\ \ f(x, y, z) = xz\, \boldsymbol{i} + xyz\, \boldsymbol{j} - \ y^2 \boldsymbol{k}, find\ div\ f$$

$$div\, f = \nabla \cdot f$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

*Div F is a scalar field.

Let us turn to the more immediate practical task of gaining a feel for the significance of the divergence. Let f(x, y, z) be a twice differentiable scalar function. Then, its gradient exists

$$\mathbf{v} = \text{grad}\, f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and we can differentiate once more, the first component with respect to *x*, the second with respect to *y*, the third with respect to *z*, and then form the divergence,

$$\text{div}\, \mathbf{v} = \text{div}\, (\text{grad}\, f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Hence, we have the basic result that *the divergence of the gradient is the Laplacian*

$$\text{div}\, (\text{grad}\, f) = \nabla^2 f.$$

Exercises

1) Find divergence from the gradient, *div (grad f)*

(a) $f = e_{xyz}$

(b) $f = z - \sqrt{x^2 + y^2}$

2) Find ***div v*** and its value at ***P***

(a) $v = x^2 i + 4y^2 + 9z^2$ *at P (-1, 0, ½)*

(b) $v = \cos xyz + \sin xyz$

3.4.4 CURL OF VECTOR FIELD

Let $v(x, y, z) = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k$ be a differentiable vector function of the Cartesian coordinates x, *y*, *z*. Then the **curl** *of the vector function* **v** or *of the vector field given by* **v** is defined by the "symbolic" determinant

$$\text{curl v} = \nabla \times v = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\delta}{\delta z} \\ v_1 & v_2 & v_3 \end{bmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) k$$

This is the formula when *x*, *y*, *z* are *right-handed.* If they are *left-handed*, the determinant has a minus sign in front. Instead of curl **v** one also uses the notation rot **v** or rotation of ***v***.

Example 3.9

Let $\mathbf{v} = [yz, \quad 3zx, \quad z] = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}$ with right-handed *x*, *y*, *z*.

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\mathbf{i} + y\mathbf{j} + (3z - z)\mathbf{k} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

Example 3.10

$$If\ \ F(x,y,z) = xz\,\boldsymbol{i} + xyz\,\boldsymbol{j} - y^2\boldsymbol{k}, find\ curl\ F$$

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\boldsymbol{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\boldsymbol{j}$$

$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\boldsymbol{k}$$

$$= (-2y - xy)\boldsymbol{i} - (0 - x)\boldsymbol{j} + (yz - 0)\boldsymbol{k}$$

$$= -y(2 + x)\,\boldsymbol{i} + x\,\boldsymbol{j} + yz\,\boldsymbol{k}$$

*Curl F is a vector field.

Grad, Div, Curl

Gradient fields are ***irrotational***. *That is, if a continuously differentiable vector function is the gradient of a scalar function f, then its curl is the zero vector,*

$$\mathrm{curl}(\mathrm{grad}\, f) = \mathbf{0}.$$

Furthermore, the divergence of the curl of a twice continuously differentiable vector function **v** *is zero,*

$$\mathrm{div}(\mathrm{curl}\,\mathbf{v}) = 0.$$

Exercises

Compute the curl of the following vector field:

a) $f(x,y,z) = < e^x \cos y\,, e^x \sin y, 0 >$

b) $f(x,y,z) = \frac{2xy}{z}\ i + xe^{xy} j + \cos(xy^2)k$

c) $f(x,y,z) = (xyz)i + (x^2 + 2yz)j + (x^2 + y^2 + z^2)k$