VECTOR ALGEBRA II

WEEK 4: VECTOR ALGEBRA II

4.1 PRODUCT (MULTIPLICATION OF TWO VECTORS)

There are two types of product of vectors.

Dot product	Cross product
- gives a scalar as the dot product of two vectors - also known as scalar/inner product	- gives a vector as the dot product of two vectors - also known as vector product

4.1.1 Dot Product

Definition

The dot product $\underset{\sim}{a}.\underset{\sim}{b}$ (read "$\underset{\sim}{a}$" dot "$\underset{\sim}{b}$") is defined by:

$$\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\, cos\, \theta \;, 0 \leq \theta \leq \pi$$

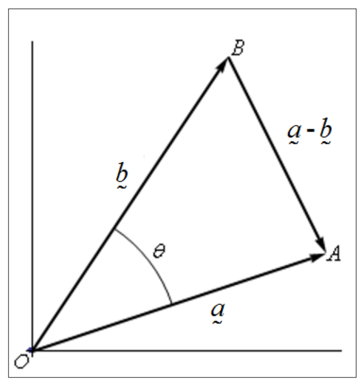
Where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$; θ is measured when the vectors have their initial point coinciding.

Proof:

Law of Cosines: $|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\, cos\, \theta$ ---- (a)

$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = (\underset{\sim}{a} - \underset{\sim}{b}).(\underset{\sim}{a} - \underset{\sim}{b})$

$\quad = |\underset{\sim}{a}|^2 - 2\underset{\sim}{a}.\underset{\sim}{b} + |\underset{\sim}{b}|^2$ ---- (b)



So when Eqn. (a) = Eqn. (b),

$$|\underset{\sim}{a}|^2 - 2\underset{\sim}{a}\underset{\sim}{b} + |\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\, cos\, \theta$$

$$\cancel{|\underset{\sim}{a}|^2} \cancel{-}2\underset{\sim}{a}\underset{\sim}{b} + \cancel{|\underset{\sim}{b}|^2} = \cancel{|\underset{\sim}{a}|^2} + \cancel{|\underset{\sim}{b}|^2} \cancel{-} 2|\underset{\sim}{a}||\underset{\sim}{b}|\, cos\, \theta$$

$$\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\, cos\, \theta$$

Properties of the dot product of vectors

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be two vectors and let α be a scalar. Then

i. $\underset{\sim}{a}.\underset{\sim}{b} = \underset{\sim}{b}.\underset{\sim}{a}$ (commutative law)

ii. $\underset{\sim}{a}.(\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{a}.\underset{\sim}{c}$ (distributive law)

iii. $\alpha(\underset{\sim}{b}.\underset{\sim}{c}) = (\alpha\underset{\sim}{b}).\underset{\sim}{c}$ or $\underset{\sim}{b}.(\alpha\underset{\sim}{c})$ where α is a scalar

iv. The dot product of two vectors (i.e., $\underset{\sim}{a}.\underset{\sim}{b}$) is a scalar.

Precaution: The dot product cannot function between scalar and vector (i.e., $3.\underset{\sim}{b}$ or $\underset{\sim}{a}.\underset{\sim}{b}.\underset{\sim}{c}$ or $\underset{\sim}{a}.\underset{\sim}{b}.\underset{\sim}{c}.\underset{\sim}{d}.\underset{\sim}{e}$)

Orthogonal vector (Also known as perpendicular or normal vector)

We have dot product, $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, orthogonal vector has $\underset{\sim}{a}.\underset{\sim}{b} = 0$

To let $\underset{\sim}{a}.\underset{\sim}{b} = 0$, (i) $\underset{\sim}{a}$ or $\underset{\sim}{b}$ are zero vectors

(ii) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are orthogonal vectors ($\underset{\sim}{a} \perp \underset{\sim}{b}$), because $\theta = 90°$; $\cos\theta = 0$

i.e., in the 3D system, $\underset{\sim}{i} \perp \underset{\sim}{j}$, $\underset{\sim}{j} \perp \underset{\sim}{k}$, $\underset{\sim}{i} \perp \underset{\sim}{k}$, therefore the dot products between them are zero (i.e., $\underset{\sim}{i}.\underset{\sim}{j} = 0$; $\underset{\sim}{j}.\underset{\sim}{k} = 0$ and $\underset{\sim}{k}.\underset{\sim}{i} = 0$)

Parallel Vector

We have dot product, $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, parallel vector has $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$

To let $\underset{\sim}{a}.\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$, we need to have $\theta = 0°$; $\cos\theta = 1$

(i) Dot product of two similar vectors $\underset{\sim}{a}.\underset{\sim}{a} = |\underset{\sim}{a}|^2$

(ii) Dot product of unit vector $\hat{\underset{\sim}{a}}.\hat{\underset{\sim}{b}} = |\hat{\underset{\sim}{a}}|^2$ or $|\hat{\underset{\sim}{b}}|^2 = 1$

i.e., in the 3D system, $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ are unit vectors, therefore the dot products between them are one (i.e., $\underset{\sim}{i}.\underset{\sim}{i} = 1$; $\underset{\sim}{j}.\underset{\sim}{j} = 1$ and $\underset{\sim}{k}.\underset{\sim}{k} = 1$)

Dot product in coordinates

Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1 \imath \underset{\sim}{i} + b_2 \jmath \underset{\sim}{i} + b_3\underset{\sim}{k}$

Then, $\underset{\sim}{a}.\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$

Proof:

$$\begin{aligned}
\underset{\sim}{a}\underset{\sim}{b} &= (a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}).(b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}) \\
&= a_1b_1(\underset{\sim}{i}.\underset{\sim}{i}) + \cancel{a_2b_1(\underset{\sim}{j}.\underset{\sim}{i})} + \cancel{a_3b_1(\underset{\sim}{k}.\underset{\sim}{i})} + \\
&\quad \cancel{a_2b_1(\underset{\sim}{j}.\underset{\sim}{i})} + a_2b_2(\underset{\sim}{j}.\underset{\sim}{j}) + \cancel{a_2b_3(\underset{\sim}{j}.\underset{\sim}{k})} + \\
&\quad \cancel{a_3b_1(\underset{\sim}{k}.\underset{\sim}{i})} + \cancel{a_3b_2(\underset{\sim}{k}.\underset{\sim}{j})} + a_3b_3(\underset{\sim}{k}.\underset{\sim}{k}) \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}$$

<u>Exercise</u>

(i) Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, what is the results for the following dot product?

(a) $\underset{\sim}{a}.\underset{\sim}{i} + \underset{\sim}{b}.\underset{\sim}{j} + \underset{\sim}{a}.\underset{\sim}{b} + a_1\underset{\sim}{b} + b_1\underset{\sim}{a}$

(b) $a_1.\underset{\sim}{i} + b_2.\underset{\sim}{j} + a_3.\underset{\sim}{k}$

(c) $\underset{\sim}{a}.\underset{\sim}{i}.a_1 + \underset{\sim}{b}.\underset{\sim}{j}.b_2 + \underset{\sim}{a}.\underset{\sim}{b}.\underset{\sim}{i}.\underset{\sim}{j} + a_1\underset{\sim}{b}.\underset{\sim}{k} + b_1\underset{\sim}{a}.\underset{\sim}{b}.\underset{\sim}{k}.\underset{\sim}{k}$

(ii) Let $\underset{\sim}{a} = \langle 1,2,3 \rangle$ and $\underset{\sim}{b} = \langle 2,0,4 \rangle$. Find

(a) $\underset{\sim}{a}.\underset{\sim}{b}$

(b) the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(iii) Let $\underset{\sim}{a} = \langle 2,-1 \rangle$ and $\underset{\sim}{b} = \langle -1/2, 1/4 \rangle$. Determine if the following vectors are parallel, orthogonal or neither.

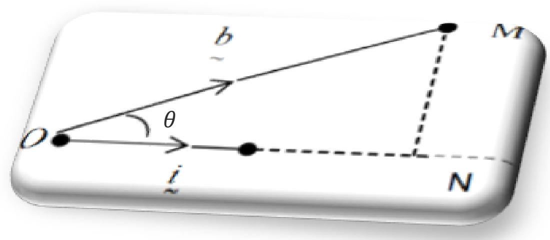
Projection of vector

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be two nonzero vectors and let $\hat{\underset{\sim}{a}}$ be the unit vector in the direction of $\underset{\sim}{a}$.

Then the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is defined as $\underset{\sim}{b}.\hat{\underset{\sim}{a}} = \underset{\sim}{b}.\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

The component of $\underset{\sim}{b}$in the direction of $\underset{\sim}{a}$ is defined as $(\underset{\sim}{b}.\hat{\underset{\sim}{a}})\hat{\underset{\sim}{a}}$

Geometric Interpretation



Let the angle between the vectors $\underset{\sim}{b}$ and $\underset{\sim}{i}$ be θ

Then, the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{i}$ = $\underset{\sim}{b}.\hat{\underset{\sim}{i}}$

$= |\underset{\sim}{b}||\hat{\underset{\sim}{i}}| \cos\theta$

$= |\underset{\sim}{b}| \cos\theta \qquad \because \ |\hat{\underset{\sim}{i}}| = 1$

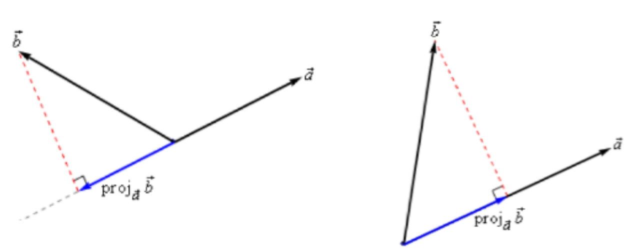
= ON = the length of the orthogonal projection of $\underset{\sim}{b}$ on a straight line parallel to $\hat{\underset{\sim}{i}}$

The component of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{i}$, $(\underset{\sim}{b}.\hat{\underset{\sim}{i}})\hat{\underset{\sim}{i}}$

$= (|\underset{\sim}{b}| \cos\theta)\underset{\sim}{i} \qquad \because \ \hat{\underset{\sim}{i}} = \underset{\sim}{i}$ (both also unit vector)

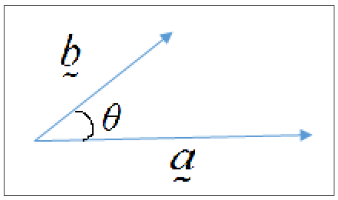
= $\overrightarrow{ON}$

Note: $\overrightarrow{ON}$ is parallel to $\underset{\sim}{i}$. The projection of $\underset{\sim}{b}$ onto $\underset{\sim}{i}$ takes the negative sign if $\overrightarrow{ON}$ is in the opposite direction of $\underset{\sim}{i}$ and vice versa.



Theorem

If $\underset{\sim}{a}$ is a given vector, then any vector $\underset{\sim}{b}$ (i.e., $\underset{\sim}{b} = \underset{\sim}{b}_1 + \underset{\sim}{b}_2$) can be expressed as the sum of a vector parallel to $\underset{\sim}{a}$ (i.e., $\underset{\sim}{b}_1 \,||\, \underset{\sim}{a}$ $\underset{\sim}{b}_1 \parallel \underset{\sim}{a}$) and a vector perpendicular to $\underset{\sim}{a}$ (i.e., $\underset{\sim}{b}_2 \parallel \underset{\sim}{a}$).



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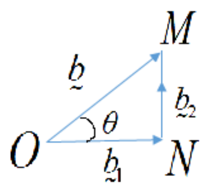


Proof

From the diagram, $\overrightarrow{OM} = \overrightarrow{ON} + \overrightarrow{NM}$ ------------------ (a)

Previously we got $\overrightarrow{ON}= (\underset{\sim}{b}.\hat{\underset{\sim}{i}})\hat{\underset{\sim}{i}}$ for projection of vector $\underset{\sim}{b}$ onto vector $\underset{\sim}{i}$

In this case, $\overrightarrow{ON}= (\underset{\sim}{b}.\hat{\underset{\sim}{a}})\hat{\underset{\sim}{a}}$ for projection of vector $\underset{\sim}{b}$ onto vector $\underset{\sim}{a}$

$\overrightarrow{ON}= (\underset{\sim}{b}.\hat{\underset{\sim}{a}})\hat{\underset{\sim}{a}} = (\frac{\underset{\sim}{b}.\underset{\sim}{a}}{|\underset{\sim}{a}|})\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = (\frac{\underset{\sim}{b}.\underset{\sim}{a}}{|\underset{\sim}{a}|^2})\underset{\sim}{a} = (\frac{\underset{\sim}{b}.\underset{\sim}{a}}{\underset{\sim}{a}.\underset{\sim}{a}})\underset{\sim}{a}$; $\overrightarrow{OM} = \underset{\sim}{b}$; ------------------ (b)

Thus, $\overrightarrow{NM} = \overrightarrow{OM} - \overrightarrow{ON} = b - \left(\frac{\underset{\sim}{b}.\underset{\sim}{a}}{\underset{\sim}{a}.\underset{\sim}{a}}\right)\underset{\sim}{a}$ ------------------ (c)

Subs. Eqns. (b) and (c) into Eqn. (a):

$$\overrightarrow{OM} = \overrightarrow{ON} + \overrightarrow{MN}$$

$$\underset{\sim}{b} = \underbrace{(\frac{\underset{\sim}{b}.\underset{\sim}{a}}{\underset{\sim}{a}.\underset{\sim}{a}})\underset{\sim}{a}}_{\underset{\sim}{b}_1 \parallel \underset{\sim}{a}} + \underbrace{\{\underset{\sim}{b} - (\frac{\underset{\sim}{b}.\underset{\sim}{a}}{\underset{\sim}{a}.\underset{\sim}{a}})\underset{\sim}{a}\}}_{\underset{\sim}{b}_2 \perp \underset{\sim}{a}}$$

<u>Exercise</u>

Let $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

(a) Find the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$

(b) Find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

(c) Express the $\underset{\sim}{b}$ as the sum of a vector parallel to $\underset{\sim}{a}$ and a vector perpendicular to $\underset{\sim}{a}$ for case (a)

(d) Express the $\underset{\sim}{a}$ as the sum of a vector parallel to $\underset{\sim}{b}$ and a vector perpendicular to $\underset{\sim}{a}$ for case (b)

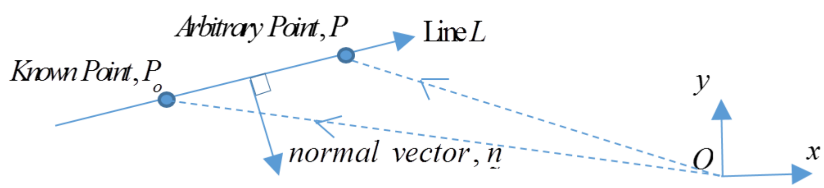
4.1.2 Applications of dot Product in Geometry

We can use dot product to find the line equation and extend it to plane equation. Besides, we will use it to find distance between point-line, point-plane, parallel-lines, parallel-planes. Furthermore, we can use it to find angles between intersecting lines and intersection planes.

(a) The equation of line (2D) and plane (3D) perpendicular to a given vector

<u>(i) Equation of line</u>

Let L be a line passing through to the point $P_0(x_0, y_0)$ and perpendicular to the vector $\underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j}$. Let $P(x, y)$ be any point on the line L.



Then the vector $\overrightarrow{P_oP}$ is along the line L and hence $\overrightarrow{P_oP} \perp \underset{\sim}{n}$

So, $(\overrightarrow{P_0P}).\underset{\sim}{n} = 0$

>> $(\overrightarrow{OP} - \overrightarrow{OP_0}).\underset{\sim}{n} = 0$ or $(\underset{\sim}{p} - \underset{\sim}{p_0}).\underset{\sim}{n} = 0$

>> $\{(x\underset{\sim}{i}, y\underset{\sim}{j}) - (x_0\underset{\sim}{i}, y_0\underset{\sim}{j})\}.(a\underset{\sim}{i} + b\underset{\sim}{j}) = 0$

>> $a(x - x_0) + b(y - y_0) = 0$

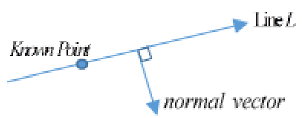
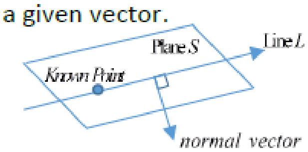
>> $ax + by = ax_0 + by_0$

>> $ax + by = c$, where $c = ax_0 + by_0$ is a scalar.

This is known as Cartesian equation of the line L (for 2D space use only).

Note: The components of $\underset{\sim}{n}$ are the coefficients of the line equation, a and b. From the Cartesian equation, we can know the information of the vector normal to the line, $\underset{\sim}{n}$.

Additional remarks: Note that the Cartesian equation of line L is restricted for plotting 2D line only. We need to use Vector or Parametric Eqn. of line L learned in Section 4.1.3 for both 2D and 3D line plotting purpose.

<table>
<tr><th>(i) Problems of 2D line in Cartesian form</th><th>(ii) Problems of 3D plane in Cartesian form</th></tr>
<tr><td>1. Find the <u>Cartesian equation of a line</u> passing through a given point and normal to a given vector

</td><td>1. Find the <u>Cartesian equation of a plane</u> passing through a given point and normal to a given vector.

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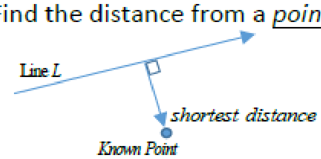
<tr><td>2. Find the distance from a <u>point to a line</u>

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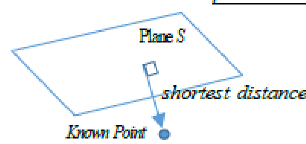
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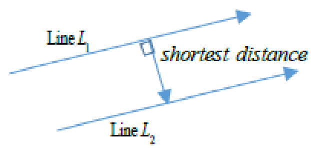
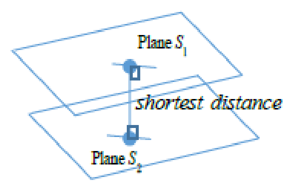
<tr><td>3. Find the distance between two <u>parallel lines</u>

</td><td>3. Find the distance between two <u>parallel planes</u>

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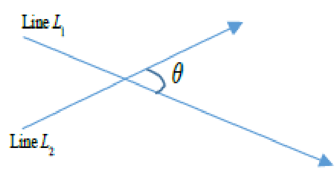
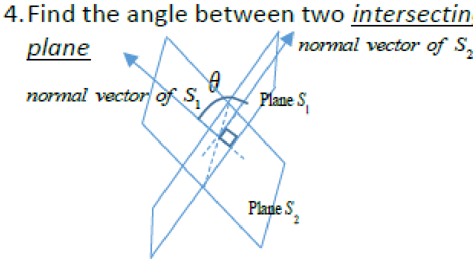
<tr><td>4. Find the angle between two <u>intersecting lines</u>

</td><td>4. Find the angle between two <u>intersecting plane</u>

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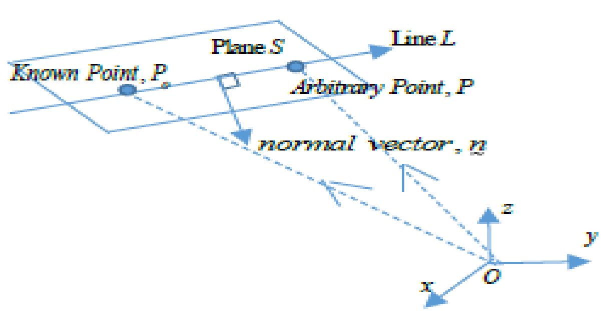
<u>(ii) Equation of Plane</u>

Let S be a plane passing through to the point $P_0(x_0, y_0, z_0)$ and normal to the vector $\underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$. Let $P(x, y, z)$ be any point in the plane S.

Then the vector $\overrightarrow{P_0P}$ is in the plane S and hence $\overrightarrow{P_0P} \perp \underset{\sim}{n}$.



So, $(\overrightarrow{P_0P}).\underset{\sim}{n} = 0$

>> $(\overrightarrow{OP} - \overrightarrow{OP_0}).\underset{\sim}{n} = 0$ or $(\underset{\sim}{p} - \underset{\sim}{p_0}).\underset{\sim}{n} = 0$

>> $\{(x\underset{\sim}{i}, y\underset{\sim}{j}, z\underset{\sim}{k}) - (x_0\underset{\sim}{i}, y_0\underset{\sim}{j}, z_0\underset{\sim}{k})\}.(a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}) = 0$

>> $a(x - x_0) + b(y - y_0) + b(z - z_0) = 0$

>> $ax + by + cz = ax_0 + by_0 + cz_0$

>> $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$ is a scalar.

This is known as Cartesian/plane equation of the plane S.

Note: the components of $\underset{\sim}{n}$ are the coefficients of the plane equation, a, b and c. From the Cartesian/plane equation, we can know the information of the vector normal to the line, $\underset{\sim}{n}$.

Precaution: You might think single equation such as ax + by + cz = d would be the general equation of a line in 3 dimensions. However, such an equation defines a **plane** in **R^3**, which geometrically is a flat surface which carries on forever in the space.

Orthogonal vectors in 2D space	Orthogonal vectors in 3D space
In 2 dimensions the orthogonal vector is unique and forms a unique 2D line.	In 3 dimensions, a vector has infinitely many orthogonal vectors, which sweep out around it forming a plane.

Think: So how is a line defined in 3 dimensions? You will learn this after knowing the cross product.

<u>Exercise:</u>

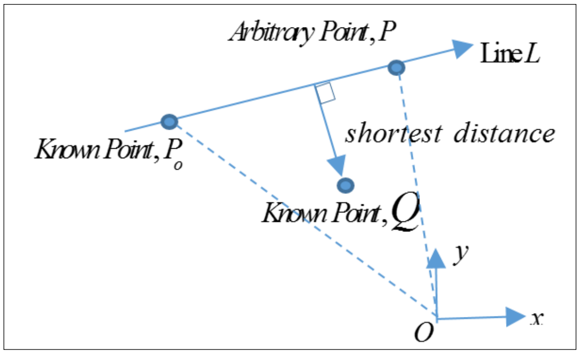
(i) Find the Cartesian equation of the line Lin the plane passing through the point $A(2,3)$ and perpendicular to the vector $\underset{\sim}{n} = \underset{\sim}{i} - 3\underset{\sim}{j}$

(ii) Find the Cartesian equation of the plane Spassing through the point $A(1,1,-1)$ and normal to the vector $\underset{\sim}{n} = -2\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$

(b) The Distance from a Point to a Line or to a Plane

(i) Distance of point-to-line

Let L be a line with the Cartesian equation$ax + by = c$ and let P be a point on the line L. From the Cartesian equation of the line L, the vector$\underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j}$ is perpendicular to L.



Previously you learnt the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ is defined as $\underset{\sim}{b}.\hat{\underset{\sim}{a}} = \underset{\sim}{b}.\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

Thus, distance of point Q to the line L = projection of the vector $\overrightarrow{PQ}$ onto the vector $\underset{\sim}{n}$

$$= \left|\overrightarrow{PQ}\,.\,\hat{\underset{\sim}{n}}\right| = \left|\overrightarrow{PQ}\,.\,\frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}\right|$$

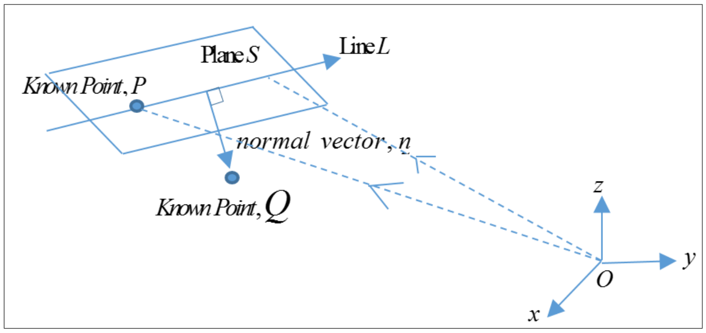
(ii) Distance of point-to-plane

Let S be a plane with the equation$ax + by + cz = d$ and let P be a point on the plane S. From the Cartesian equation of the plane S, the vector$\underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ is normal to S.

Then, the distance from the point Q to the plane S is the projection of the vector $\overrightarrow{PQ}$ onto the vector $\underset{\sim}{n}$.

Distance point Q to the plane S

$$= \left|\overrightarrow{PQ}\,.\,\hat{\underset{\sim}{n}}\right| = \left|\overrightarrow{PQ}\,.\,\frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}\right|$$



Exercise:

(i) Find the distance from the point Q $(4,4)$ to the line $L : x + 3y = 6$

(ii) Find the distance from the point Q $(3,2,-1)$ to the plane S: $-2x + 3y - z = 2$

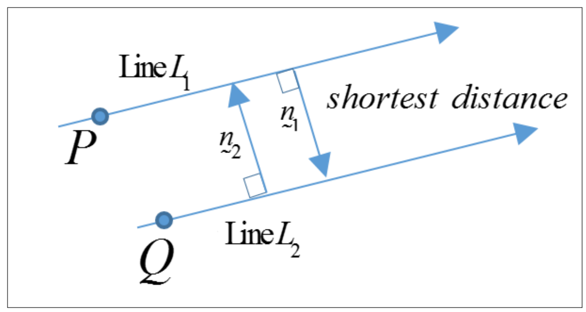
(c) Distance between two parallel lines or two parallel planes

(i) <u>Distance of two-parallel-lines</u>

The projection method is also used to find the distance between two parallel lines L_1 and L_2. Let us choose one point P from the line L_1 and another point Q from the line L_2.

For two parallel lines, the perpendicular vectors, $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ for lines L_1 and L_2 are equal: $\underset{\sim}{n}_1 = \underset{\sim}{n}_2 = \underset{\sim}{n}$.

Distance between parallel lines L_1 and L_2= $\left|\overrightarrow{PQ} \cdot \frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}\right|$

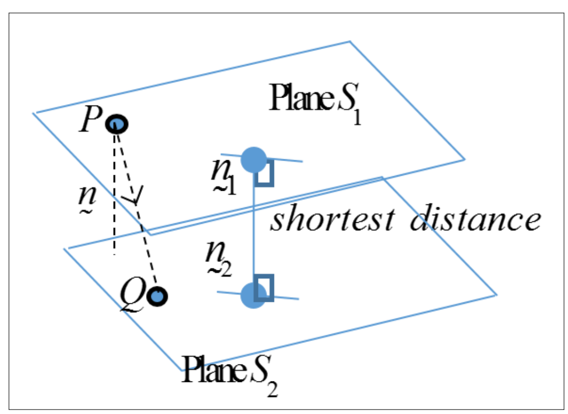


(ii) <u>Distance of two-parallel-planes</u>

The projection method is also used to find the distance between two parallel planes S_1 and S_2. Let us choose one point P from the plane S_1 and another point Q from the plane S_2.

For two parallel planes, the perpendicular vectors, $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ for planes S_1 and S_2 are equal: $\underset{\sim}{n}_1 = \underset{\sim}{n}_2 = \underset{\sim}{n}$.

Distance between parallel planes S_1 and S_2= $\left|\overrightarrow{PQ} \cdot \frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}\right|$



Exercise:

(i) Find the distance between line $L_1: x + 3y = -2$ to the line $L_1: x + 3y = 6$

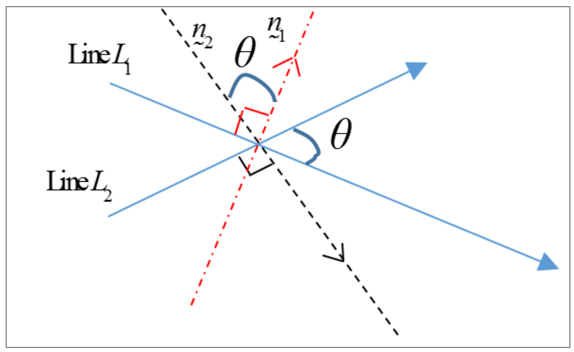
(ii) Find the distance between plane $S_1: -2x + 3y - z = 2$ to the plane $S_2: 2x - 3y + z = -15$

(iii) Find the constant distance between plane $S_1: -2x - 3y - z = 2$ to the plane $S_2: -2x + 3y - z = 15$ if exist/possible.

(d) Find the angle between two intersecting lines or two intersecting planes

(i) Angle of two-intersecting-lines

Let L_1 and L_2 be two lines with the perpendicular vectors, $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$, respectively. If L_1 and L_2 intersect, then the angle between L_1 and L_2 is equal to angle between $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$.



Therefore, $\underset{\sim}{n}_1 . \underset{\sim}{n}_2 = \left|\underset{\sim}{n}_1\right| \left|\underset{\sim}{n}_2\right| \cos\theta$

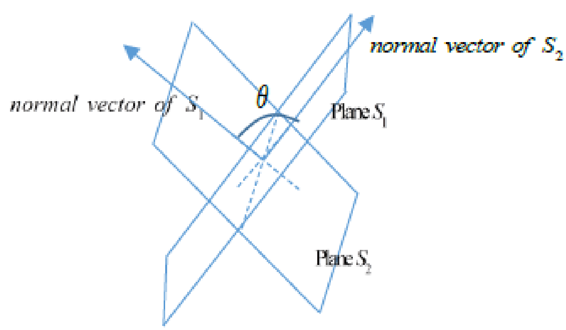
$$\theta = \cos^{-1} \frac{\underset{\sim}{n}_1 . \underset{\sim}{n}_2}{\left|\underset{\sim}{n}_1\right| \left|\underset{\sim}{n}_2\right|}$$

(ii) Angle of two-intersecting-planes

Let S_1 and S_2 be two planes with the perpendicular vectors, $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$, respectively. If S_1 and S_2 intersect, then the angle between S_1 and S_2 is equal to angle between $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$.

Therefore, $\underset{\sim}{n}_1 . \underset{\sim}{n}_2 = \left|\underset{\sim}{n}_1\right| \left|\underset{\sim}{n}_2\right| \cos\theta$

$$\theta = \cos^{-1} \frac{\underset{\sim}{n}_1 . \underset{\sim}{n}_2}{\left|\underset{\sim}{n}_1\right| \left|\underset{\sim}{n}_2\right|}$$



Exercise:

(i) Find the angle between the lines $L_1: 3x - 6y = 15$ and $L_2: 2x + y = 5$

(ii) Find the angle between the planes $S_1 : 3x - 6y - 2z = 15$ and $S_2: 2x + y - 2z = 5$

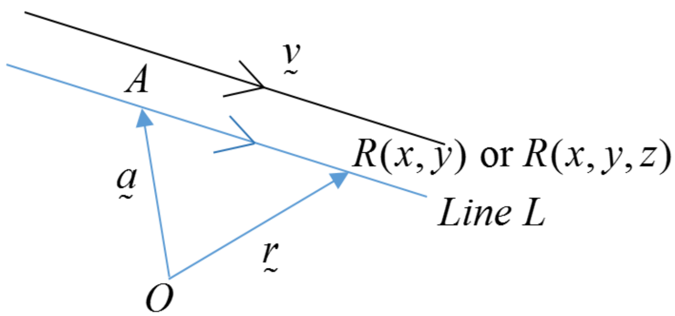
4.1.3 Application in geometry: Equations of lines in 2D and 3D spaces

Problems in space (in 2D and 3D space)

1. Find the *equation of a line* passing through a given point and parallel to a given vector (all in 2D & 3D space).
2. Determine whether two lines intersect in three dimension space and find the *point of intersection* if they intersect.

(i) Definition (Equation of a line in 2D or 3D space)

Let L be a straight line passing through the point A and is parallel to a given vector $\underset{\sim}{v}$ (i.e.,$\overrightarrow{AR} \,||\, \underset{\sim}{v}$). Suppose that $R(x,y)$ or $R(x,y,z)$ is any point on L. Find the vector equation, parametric equation and Cartesian equation of line L.



(a) Vector equation of line

(i) Let $\left(\overrightarrow{OA}\right) = \underset{\sim}{a}$ and $\left(\overrightarrow{OR}\right) = \underset{\sim}{r}$ be the position vectors of A and R respectively.

(ii) Since$\left(\overrightarrow{AR} \,||\, \underset{\sim}{v}\right)$, then$\left(\overrightarrow{AR}\right) = t\underset{\sim}{v}$, where $t \in \mathfrak{R}$.

Note: as t changes, we have all the points on the line L.

(iii) Now following *head-to-tail method*, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

Then, we get $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v}$ -- (1)

This is called the **vector equation of the line** L.

The vector $\underset{\sim}{v}$ is called a direction vector of the line L.

Note: The Eqn. (1) can be applied for problem in 2D or 3D space. Since the computation and derivation for 2D and 3D space are similar. We will give the example in 3D one for the demonstration.

(b) Parametric equation of a line in 2D and 3D space

Now, let position vector of an arbitrary point on line L, $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, position vector of a point passing through line L, $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and direction vector parallel to line L, $\underset{\sim}{v} = v_1\underset{\sim}{i} + v_2\underset{\sim}{j} + v_3\underset{\sim}{k}$

From Eqn. (1) we have

$\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v}$ >> $\langle x, y, z\rangle = \langle a_1, a_2, a_3\rangle + t\langle v_1, v_2, v_3\rangle$

>> $\underset{Point\ at\ time\ t}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{Initial\ Point}{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}} + \underset{Direction\ Vector}{t\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}}$ (Matrix form of vector equation of line)

By comparing the components of $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$, we have

$\underset{\sim}{r} = \langle x, y, z\rangle$

$$\left.\begin{aligned} x &= a_1 + tv_1 \\ y &= a_2 + tv_2 \\ z &= a_3 + tv_3 \end{aligned}\right\}, t \in \mathbb{R} \quad \ldots\ldots\ldots\ldots (2)$$

This system of equations (2) is called the **parametric equations of the line** L.

The variable/scalar $\underset{\sim}{t}$ is called the parameter of the system of equations.

Note: The Eqn. (2) can be applied for problem in 2D or 3D space in the same manner.

(c) Cartesian equation of a line in 2D and 3D space

By equal the parameter t in the Eq. (2), we have

$$t = \frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3} \quad \ldots\ldots\ldots\ldots (3)$$

The Eqn. (3) is called the **Cartesian equation or symmetric form of the line** L.

Precaution Zero denominator leads to zero numerator as shown in Eqn. (3).

Proof: From Eqn. (2), if $v_1 = 0$, then $x - a_1 = 0$. This also apply to v_2 and v_3.

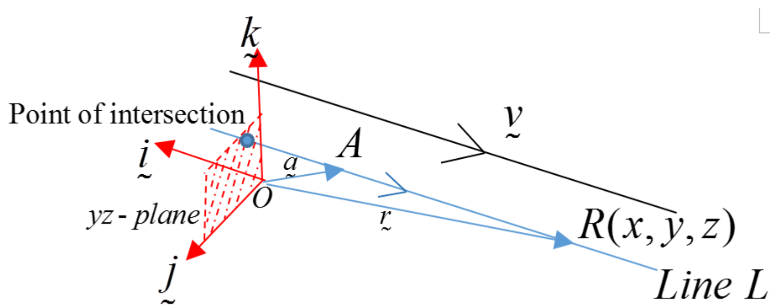
(ii) Intersection between a line to a plane or between two lines

(a) Intersection of a line to a plane

If we know the information of a plane and the line equation are given either in format of Vector Eqn./Parametric Eqn./Cartesian Eqn., for example:

Information of a plane

(i) The line intersects at a specific plane, i.e., yz-plane at coordinate $(x, y, z) = (0, y, z)$



Let say Parametric Eqn. have been derived from a given information of point A passing through the line, and the vector direction of the line L, v are given as well. Then, we can use Parametric Eqn. $\left.\begin{array}{l} x = a_1 + tv_1 \\ y = a_2 + tv_2 \\ z = a_3 + tv_2 \end{array}\right\}$, $t \in \Re$ to solve for t,y and z.

Then the point of intersection at yz-plane, $(0, y, z)$ can be obtained.

(ii) The plane of eqn. of the intersection plane is given, i.e.,$ax + by + cz = k$

The step is similar to procedure above. Given the a, b, c and k, substitute the Parametric Eqn. into the eqn. of intersection to solve for t. Then, you can get the point of intersection (x, y, z) by substitute t into Parametric Eqn.

Precaution: There are three possibilities of the intersection: (i) line intersects the plane in a point; (ii) line is parallel to the plane (no point of intersection); (iii) line is in the plane.

Note: You will know how to derive the equation of plane after you learn about ***product of vectors***.

(b) Intersection between two lines

Let us have line $L_1 : \underset{\sim}{r}_1 = \langle x_1, y_1, z_1 \rangle = \langle a_1, a_2, a_3 \rangle + t\langle v_1, v_2, v_3 \rangle$ and line $L_2 : \underset{\sim}{r}_2 = \langle x_2, y_2, z_2 \rangle = \langle b_1, b_2, b_3 \rangle + s\langle u_1, u_2, u_3 \rangle$ where t and s are the parameters, $\underset{\sim}{a} = \langle a_1, a_2, a_3 \rangle$, $\underset{\sim}{b} = \langle b_1, b_2, b_3 \rangle$ are the position vectors specified at line L_1 and L_2 respectively. $\underset{\sim}{v} = \langle v_1, v_2, v_3 \rangle$,$\underset{\sim}{u} = \langle u_1, u_2, u_3 \rangle$ are the vectors parallel to line L_1 and L_2.

If the line L_1 and line L_2 intersect each other, then:

$\underset{\sim}{r}_1 = \underset{\sim}{r}_2$

$x_1 = x_2 >> a_1 + tv_1 = b_1 + su_1$ -- (a)

$y_1 = y_2 >> a_2 + tv_2 = b_2 + su_2$-- (b)

$z_1 = z_2 >> a_3 + tv_3 = b_3 + su_3$ -- (c)

This means that all the three Eqns. (a), (b) and (c) must be satisfied if the two lines L_1 and L_1 are intersecting with each other. In the other words, if the parameter t obtained from Eqn. (a) and parameter s obtained from Eqn. (b) will not satisfy Eqn.(c) if there is no point of intersection.

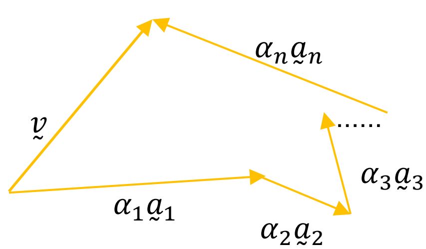
It intersection exist, the point of intersection is as following:

$(\boldsymbol{x_1 = x_2, y_1 = y_2, z_1 = z_2})$ **or** $((\boldsymbol{a_1 + tv_1}), (\boldsymbol{a_2 + tv_2}), (\boldsymbol{a_3 + tv_3}))$ **or** $(b_1 + su_1, b_2 + su_2, b_3 + su_3)$.

<u>(iii) Linear combination and linear dependence</u>

(a) Linear combination

A linear combination of two or more vectors is the vector obtained by adding two or more vectors (with different directions) which are multiplied by scalar values.



$\underset{\sim}{v} = \alpha_1 \underset{\sim}{a}_1 + \alpha_2 \underset{\sim}{a}_2 + \alpha_3 \underset{\sim}{a}_3 + \ldots + \alpha_n \underset{\sim}{a}_n$

(b) Linear dependent

Vectors are linearly dependent if there is a linear combination of them that equals the zero vector, without the coefficients of the linear combination being zero.

$\alpha_1 \underset{\sim}{a}_1 + \alpha_2 \underset{\sim}{a}_2 + \alpha_3 \underset{\sim}{a}_3 + \ldots + \alpha_n \underset{\sim}{a}_n = \underset{\sim}{0}$, where scalar $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \neq 0$

Note: The vectors are linearly dependent if the determinant of the matrix is zero, meaning that the rank of the matrix is less than its full rank.

(Hint: In a matrix system, zero determinant helps to indicate an infinite or no solution system)

$$|\underset{\sim}{a}_1 + \underset{\sim}{a}_2 + \underset{\sim}{a}_3 + \ldots + \underset{\sim}{a}_n| = 0 \; |\underset{\sim}{a}_1 + \underset{\sim}{a}_2 + \underset{\sim}{a}_3 + \ldots + \underset{\sim}{a}_n| \neq 0$$

(c) Linear independent

Vectors are linearly independent if none of them can be expressed as a combination of others

$\alpha_1 \underset{\sim}{a}_1 + \alpha_2 \underset{\sim}{a}_2 + \alpha_3 \underset{\sim}{a}_3 + \ldots + \alpha_n \underset{\sim}{a}_n = \underset{\sim}{0}$, where scalar $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \neq 0$

Note: The vectors are linearly dependent if the determinant of the matrix is non-zero, meaning that the rank of the matrix is equal to its full rank.

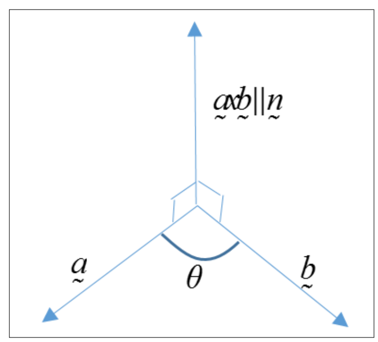
(Hint: In a matrix system, non-zero determinant helps to indicate a unique solution system)

4.1.4 Cross Product

(a) Definition

The cross product $\underset{\sim}{a} x \underset{\sim}{b}$ (read "a" cross "b") of two nonzero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$is defined by

$$\underset{\sim}{a} x \underset{\sim}{b} = \underbrace{|\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta}_{scalar} \; \underset{\sim}{n}$$



where
(i) the angle θ between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(ii) vector $\underset{\sim}{n}$ is parallel to direction of $\underset{\sim}{a} x \underset{\sim}{b}$ and it is perpendicular to both vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (i.e. $\underset{\sim}{a} x \underset{\sim}{b} \perp \underset{\sim}{a}$ and and $\underset{\sim}{a} x \underset{\sim}{b} \perp \underset{\sim}{b}$)

(b) Properties of the cross product of vectors

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be two vectors and let α be a scalar. Then

i. $\underset{\sim}{a} x \underset{\sim}{b} = -\underset{\sim}{b} x \underset{\sim}{a}$ (**Anti**-commutative Law)
ii. $\underset{\sim}{a} x(\underset{\sim}{b} + \underset{\sim}{c}) = (\underset{\sim}{a} x \underset{\sim}{b}) + (\underset{\sim}{a} x \underset{\sim}{c})$ (Distributive Law)
iii. $\alpha(\underset{\sim}{a} x \underset{\sim}{b}) = (\alpha \underset{\sim}{a}) x \underset{\sim}{b} = \underset{\sim}{a} x(\alpha \underset{\sim}{b})$ where α is a scalar
iv. The cross product of two vectors (i.e., $\underset{\sim}{a} x \underset{\sim}{b}$) is a vector.

Precaution: The cross product cannot function between scalar and vector (i.e., $3 \; x \; \underset{\sim}{b}$ or $(\underset{\sim}{a}.\underset{\sim}{b}) \; x \; \underset{\sim}{c}$ or $(\underset{\sim}{a}.\underset{\sim}{b}).(\underset{\sim}{c}.\underset{\sim}{d}) \; x \; (\underset{\sim}{e})$

(c) Orthogonal vector (Also known as perpendicular or normal vector)

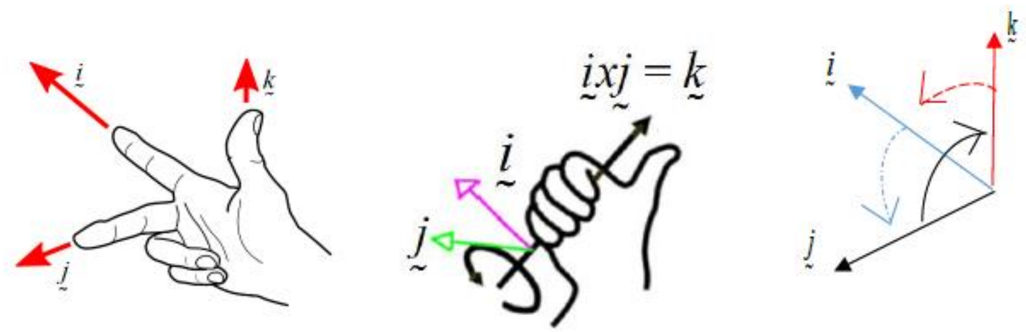
We have cross product,$\underset{\sim}{a} x \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \; \underset{\sim}{n}$, orthogonal vector has $\underset{\sim}{a} x \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \; \underset{\sim}{n}$

To let $\underset{\sim}{a} x \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\underset{\sim}{n}$, we need to have $\theta = 90°; \sin\theta = 1$

(i) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are orthogonal vectors ($\underset{\sim}{a} \perp \underset{\sim}{b}$)

(ii) Cross product of unit vector $\hat{\underset{\sim}{a}} x \hat{\underset{\sim}{b}} = |\hat{\underset{\sim}{a}}||\hat{\underset{\sim}{b}}|\underset{\sim}{n} = \underset{\sim}{n}$ *(where mag. of unit vector =1)*

i.e., in the 3D system, $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ are unit vectors where $\underset{\sim}{i} \perp \underset{\sim}{j}, \underset{\sim}{j} \perp \underset{\sim}{k}, \underset{\sim}{i} \perp \underset{\sim}{k}$, therefore the cross product between them are the vector normal to its plane (i.e., $\underset{\sim}{i} x \underset{\sim}{j} = \underset{\sim}{k}$; $\underset{\sim}{j} x \underset{\sim}{k} = \underset{\sim}{i}$ and $\underset{\sim}{k} x \underset{\sim}{i} = \underset{\sim}{j}$). This follows right hand rule and system as following:



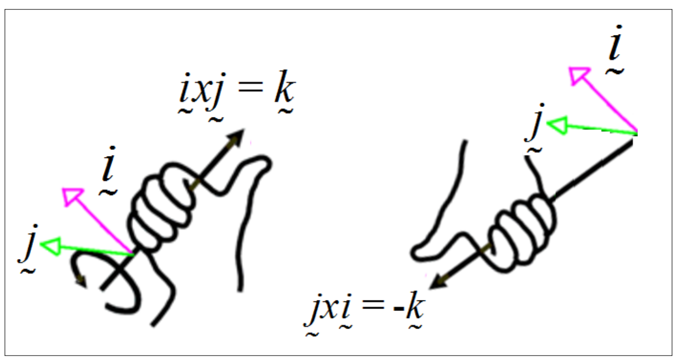
Precaution: The cross product of vectors does not satisfy the commutative law: $\underset{\sim}{a} x \underset{\sim}{b} \neq \underset{\sim}{b} x \underset{\sim}{a}$

Proof:

$\underset{\sim}{i} x \underset{\sim}{j} = \underset{\sim}{k}$

$\underset{\sim}{j} x \underset{\sim}{i} = -\underset{\sim}{k}$

Thus, $\underset{\sim}{i} x \underset{\sim}{j} = -\underset{\sim}{j} x \underset{\sim}{i}$



Apply this to other axis we get:

$\underset{\sim}{j} x \underset{\sim}{k} = \underset{\sim}{i}$ or $-\underset{\sim}{k} x \underset{\sim}{j}$
$\underset{\sim}{k} x \underset{\sim}{i} = \underset{\sim}{j}$ or $-\underset{\sim}{i} x \underset{\sim}{k}$

Remark:

i. $\underset{\sim}{a} x \underset{\sim}{b} = -\underset{\sim}{b} x \underset{\sim}{a}$
ii. $(\underset{\sim}{a} x \underset{\sim}{b}) x \underset{\sim}{c} \neq \underset{\sim}{a} x (\underset{\sim}{b} x \underset{\sim}{c})$

(d) Parallel vector

We have cross product, $\underset{\sim}{a} x \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \ \underset{\sim}{n}$, parallel vector has $\underset{\sim}{a} x \underset{\sim}{b} = \underset{\sim}{0}$ where $\underset{\sim}{a}$ & $\underset{\sim}{b} \neq \underset{\sim}{0}$

To let $\underset{\sim}{a} x \underset{\sim}{b} = \underset{\sim}{0}$, we need to have $\theta = 0°$ or $180°$; $\sin\theta = 0$

(i) Cross product of two similar vectors $\underset{\sim}{a} x \underset{\sim}{a} = \underset{\sim}{0}$ ---- (if $\underset{\sim}{a} = \underset{\sim}{b}$ $\because \theta = 0$)

(ii) Cross product of two parallel vectors $\underset{\sim}{a} x \underset{\sim}{b} = \underset{\sim}{0}$ ----- (if $\underset{\sim}{a} \parallel \underset{\sim}{b}$ $\because \theta = 0$ or π)

(iii) Cross product of two parallel unit vectors $\hat{\underset{\sim}{a}} x \hat{\underset{\sim}{b}} = \underset{\sim}{0}$ ---- (if $\hat{\underset{\sim}{a}} \parallel \hat{\underset{\sim}{b}} \because \theta = 0 \text{ or } \pi$)

i.e., in the 3D system, $\underset{\sim}{i} \parallel \underset{\sim}{i}$ & $-\underset{\sim}{i}$, $\underset{\sim}{j} \parallel \underset{\sim}{j}$ & $-\underset{\sim}{j}$, $\underset{\sim}{k} \parallel \underset{\sim}{k}$ & $-\underset{\sim}{k}$, therefore the cross product between them are zero (i.e., $\underset{\sim}{i} x \underset{\sim}{i} = 0$; $\underset{\sim}{j} x \underset{\sim}{j} = 0$ and $\underset{\sim}{k} x \underset{\sim}{k} = 0$)

Exercise:

(i) Does (a) What are the reason for $(\underset{\sim}{a} x \underset{\sim}{b}) = 0$?

(b) $\underset{\sim}{a} x \underset{\sim}{b} = \underset{\sim}{a} x \underset{\sim}{c}$

>> (If vectors $\underset{\sim}{a} \neq 0, \underset{\sim}{b} \neq \underset{\sim}{c}, \underset{\sim}{a} \neq (\underset{\sim}{b} - \underset{\sim}{c})$ find the relationship between these vectors)

(e) Cross product in coordinates

Let $\underset{\sim}{a}$= $(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k})$ and $\underset{\sim}{b}$= $(b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k})$

Then $(\underset{\sim}{a} x \underset{\sim}{b}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\underset{\sim}{i} - (a_1b_3 - a_3b_1)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k}$

Proof:

$$(\underset{\sim}{a} x \underset{\sim}{b}) = (a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}) x (b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k})$$

$$= \cancel{(a_1b_1)\underset{\sim}{i} x \underset{\sim}{i}} + (a_1b_2)\underset{\sim}{i} x \underset{\sim}{j} + (a_1b_3)\underset{\sim}{i} x \underset{\sim}{k} +$$

$$(a_2b_1)\underset{\sim}{j} x \underset{\sim}{i} + \cancel{(a_2b_2)\underset{\sim}{j} x \underset{\sim}{j}} + (a_2b_3)\underset{\sim}{j} x \underset{\sim}{k} +$$

$$(a_3b_1)\underset{\sim}{k} x \underset{\sim}{i} + (a_3b_2)\underset{\sim}{k} x \underset{\sim}{j} + \cancel{(a_3b_3)\underset{\sim}{k} x \underset{\sim}{k}} +$$

$$= \underset{\sim}{i}(a_2b_3 - a_3b_2) + \underset{\sim}{j}(a_3b_1 - a_1b_3) + \underset{\sim}{k}(a_1b_2 - a_2b_1)$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

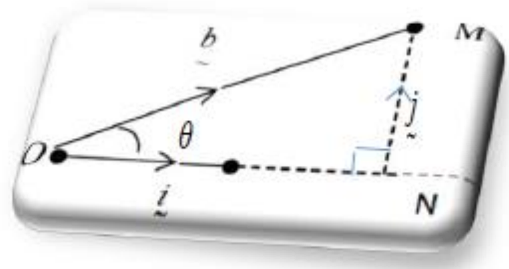
(f) Projection of vector

Previously we learnt that for the projection of vector $\underset{\sim}{b}$ onto $\underset{\sim}{i}$ using ***dot product***, $\underset{\sim}{b}.\hat{\underset{\sim}{i}}$

$= |\underset{\sim}{b}||\hat{\underset{\sim}{i}}|\cos\theta$

$= |\underset{\sim}{b}|\cos\theta \qquad \because \ |\hat{\underset{\sim}{i}}| = 1$

$= ON$ = the length of the orthogonal projection of $\underset{\sim}{b}$ on a straight line parallel to $\hat{\underset{\sim}{i}}$



Now we **extend** it for the projection of vector $\underset{\sim}{b}$ onto $\underset{\sim}{j}$ where $\underset{\sim}{j} \perp \underset{\sim}{i}$

Then, the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{j}$ = $\underset{\sim}{b}.\hat{\underset{\sim}{j}}$

$= |\underset{\sim}{b}||\hat{\underset{\sim}{j}}|\,sin\,\theta$

$= |\underset{\sim}{b}|\,sin\,\theta \qquad \because \ |\hat{\underset{\sim}{j}}| = 1$ -- (1)

$= NM$ = the length of the orthogonal projection of $\underset{\sim}{b}$ on a straight line perpendicular to $\hat{\underset{\sim}{i}}$

The component of $\underset{\sim}{b}$in the direction of $\underset{\sim}{j}$, $(\underset{\sim}{b}.\hat{\underset{\sim}{j}})\hat{\underset{\sim}{j}}$

$= (|\underset{\sim}{b}|\sin\theta)\underset{\sim}{j} \qquad \because \ \hat{\underset{\sim}{j}} = \underset{\sim}{j}$ (both also unit vector) -- (2)

= $\overrightarrow{NM}$

Now we look at the ***cross product*** of a vector and a unit vector, then we check its magnitude:

Cross product, $\underset{\sim}{b}x\hat{\underset{\sim}{i}} = \underbrace{|\underset{\sim}{b}||\hat{\underset{\sim}{i}}|\sin\theta}_{scalar}\ \underset{\sim}{n}$

Precaution: The normal vector, $\underset{\sim}{n}$ is not $\underset{\sim}{j}$. You will learn the component of $\underset{\sim}{b}$in the direction of $\underset{\sim}{j}$ using cross product after you learn the triple vector product later.

Magnitude, $|\underset{\sim}{b}x\hat{\underset{\sim}{i}}| = ||\underset{\sim}{b}|\sin\theta|$ ---(3)

$\therefore$ Combining Eqns. (1) & (3), we get the magnitude $\underset{Cross}{|\underset{\sim}{b}x\hat{\underset{\sim}{i}}|} = ||\underset{\sim}{b}|\sin\theta| = \underset{Dot}{|\underset{\sim}{b}.\hat{\underset{\sim}{j}}|}$

As we know the range of angle is $0 \le \theta \le \pi$, $sin\,\theta > 0$

We can simplify it into:

$\underset{Cross}{|\underset{\sim}{b}x\hat{\underset{\sim}{i}}|} = |\underset{\sim}{b}|\sin\theta = \underset{Dot}{\underset{\sim}{b}.\hat{\underset{\sim}{j}}}$

It indicates that the length of NM (i.e., the distance from point $N$to point M or vice versa) can be found from:

(i) Dot product (i.e., $\underset{\sim}{b}.\underset{\sim}{\hat{j}}$)

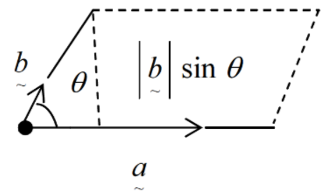
(ii) Cross product (i.e., $|\underset{\sim}{b}x\underset{\sim}{\hat{i}}|$)

This is very useful in the geometry application especially to find the shortest distance between point, line, or plane.

Projection of vector – Area of Parallelogram

The projection of vector is useful in finding the area of parallelogram as shown in figure below:

Vector $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are represented by two sides of a parallelogram



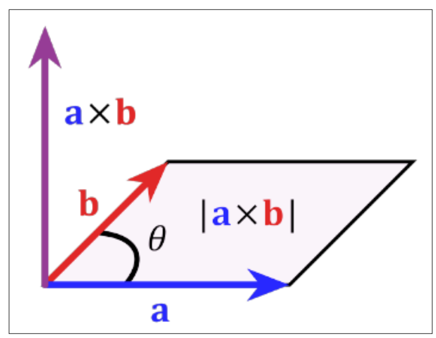
Area of parallelogram

= Height of parallelogram x Length

$= (|\underset{\sim}{b}|\, sin\,\theta)\; x\; (|\underset{\sim}{a}|)$

$= |\underset{\sim}{a}||\underset{\sim}{b}|\, sin\,\theta$---(1)

Relationship – Area of Parallelogram & Cross Product



From cross product of vector $\underset{\sim}{a}$ and $\underset{\sim}{b}$, we have

$\underset{\sim}{a}x\underset{\sim}{b} = \underbrace{|\underset{\sim}{a}||\underset{\sim}{b}|\, sin\,\theta}_{scalar}\; \underset{\sim}{n}$

The magnitude of the cross product is

$|\underset{\sim}{a}x\underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|\, sin\,\theta$-- (2)

Note: Eqn. (1) = Eqn. (2) denotes that the magnitude of cross product is equal to the area of its area of parallelogram.

Area of parallelogram = Height of parallelogram x Length $= |\underset{\sim}{a}||\underset{\sim}{b}|\, sin\,\theta = |\underset{\sim}{a}x\underset{\sim}{b}|$----------------- (3)

Area of triangle $= \frac{1}{2}x$ Height of parallelogram x Length $= \frac{1}{2}|\underset{\sim}{a}x\underset{\sim}{b}|$-------------------------------- (4)

Exercise:

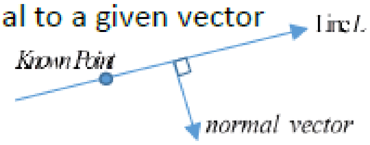
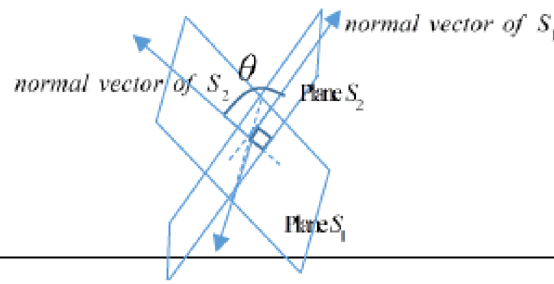
(ii) Let $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$ be three points.

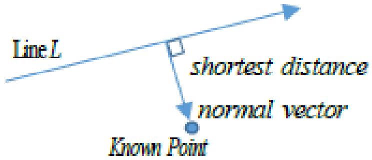
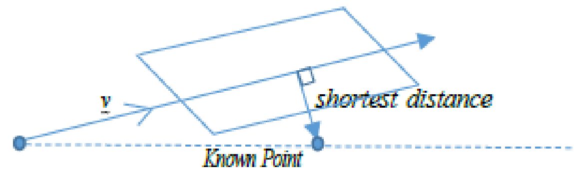
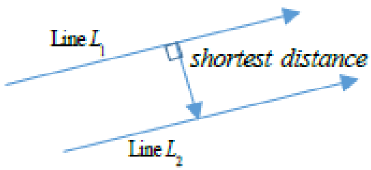
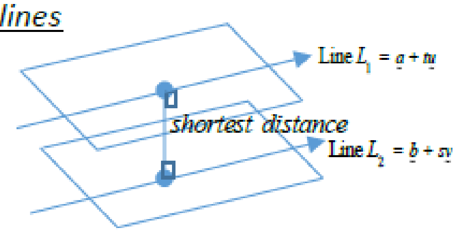
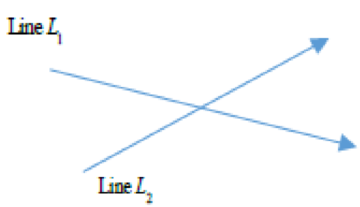
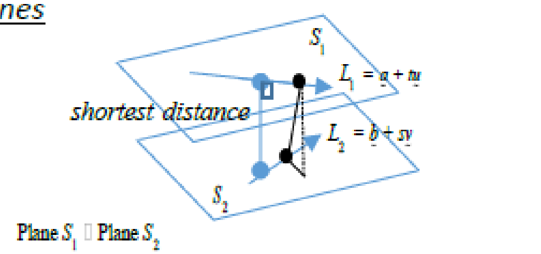
Find (a) a vector normal the plane containing P, $Q$and R

(b) the area of the triangle P, $Q$and R.

4.1.5 Application of Cross Product in Geometry (given Parallel Vector)

Previous dot product applications are used mainly when the normal vector is given. When the parallel vector is given, we can use the cross product to find the 3D-line equation between intersecting planes, distance between point-to-3D-line, distance between two 3D-parallel-lines, and distance between two 3D-skew-lines.

(i) Previous Problems of 2D line in general Cartesian form	**(ii) Current Problems of 3D line in Parametric form**
1. Find the <u>general *Cartesian equation of a line*</u> passing through a given point and normal to a given vector 	1. Find the <u>*parametric equation of a 3D line*</u> form by two intersecting planes. 

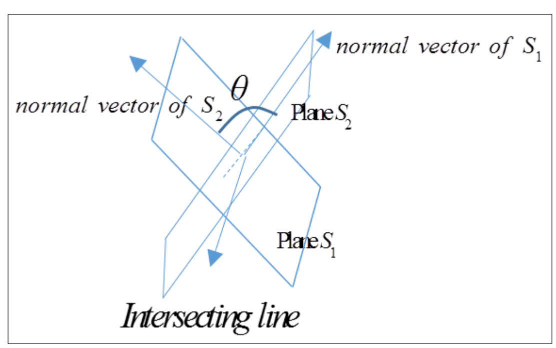
2. Find the distance from a *point to a line* 	2. Find the distance from a *point to 3D line* 
3. Find the distance between two *parallel lines* 	3. Find the distance between two *3D parallel lines* 
4. Find constant distance between two *skew lines – NOT EXIST!* 	4. Find the distance between two *3D skew lines* 

Recall & Motivation: Previously we learned to define a line of equation of a 2D line using general Cartesian Eqn (involve dot product). Now, we have learned the cross product operation, and this enable you to define a line of equation of a 3D line using Parametric Eqn.

Idea: 3D line if formed where two planes intersect.

(a) Parametric equation of 3D line

Let S_1 and S_2 be two planes with normal vectors $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ respectively. If S_1 and S_2 intersect, then the intersection is a line L. Since L is in both S_1 and S_2, L is normal to both S_1 and S_2, L is normal both $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$.



Therefore, $L \| \left(\underset{\sim}{n_1} \, x \, \underset{\sim}{n_2}\right)$

Let P be a point on a line L (i.e. P is in both S_1 and S_2)

Then the vector equation of L is given by

$$L: r = OP + t\left(\underset{\sim}{n_1} \, x \, \underset{\sim}{n_2}\right)$$

Exercise:

(i) Find the parametric equations of the intersecting line L of the planes

$S_1: 3x - 6y - 2z = 15$ --- (a)

$S_2: 2x + y - 2z = 5$ --- (b)

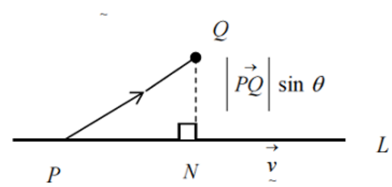
(ii) Describe the procedure to plot the 3D line L by using the parametric equations obtained in *Ans (i)* . Can we know the direction of the line L from the plotting by using the parametric equation?

(iii) Generated the general Cartesian eqn. from the Parametric eqn. of 3D line L . What are the major advantage of having general Cartesian line eqn. Can we know the direction of the line L from the plotting by using the general Cartesian equation?

(b) Distance of point-to-3D-line

Let *L* be a line in a space passing through the point *P* with the vector equation $\underset{\sim}{r} = \underset{\sim}{p} + t\underset{\sim}{v}, t \in \mathbb{R}$

By using projection method and cross method as shown in Section 4.1.4 (f), we know the distance from the point *Q* to the line *L*



$= \left|\left|\overrightarrow{PQ}\right| \sin\theta\right|$ where θ is angle between $\left|\overrightarrow{PQ}\right|$ and $\underset{\sim}{v}$.

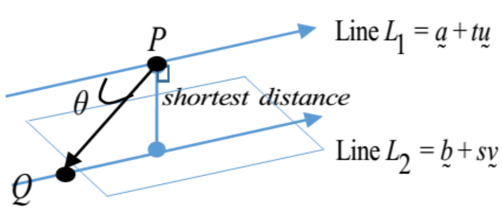
$= \left|\overrightarrow{PQ} \times \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}\right|$ $\because \left|\overrightarrow{PQ} \times \underset{\sim}{v}\right| = \left|\overrightarrow{PQ}\right||\underset{\sim}{v}| \sin\theta = \left|\overrightarrow{PQ}\right| \sin\theta$

Exercise

Find the distance from the point Q (1,1,5) to the line *L* with the parametric equations $x = 1 + t,\ y = 3 - t,$ $z = 2t.$

(c) Distance of two-parallel-3D-lines

The above method is also used to find the distance between two parallel lines L_1 and L_2 in space (They have the same direction vector $\underset{\sim}{v} = \underset{\sim}{u}$). In this case, we choose one point P from L_1 and another point Q from L_2.



By using projection method and cross method as shown in Section 4.1.4(f), we know the distance from the point Q to the line L.

$\left|\left|\overrightarrow{PQ}\right| \sin\theta\right| = \left|\overrightarrow{PQ} \times \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}\right|$ or $\left|\overrightarrow{PQ} \times \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}\right|$

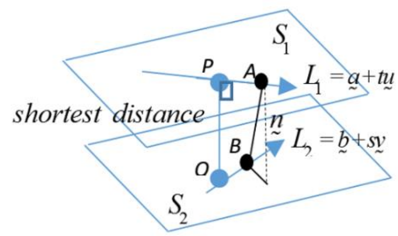
Exercise

Find the distance from the two line L_1 with the parametric equations: $x = 1 + t,\ y = 3 - t,\ z = 2t$; and with the vector equation $\underset{\sim}{r} = \langle 5,2,-1\rangle + t\langle 1,-1,2\rangle$

(d) Distance between two 3D skew lines

Let L_1 and L_2 be two skew lines in a 3D space with the vector equations $L_1: \underset{\sim}{a} + t\underset{\sim}{u}$ and $L_2: \underset{\sim}{b} + s\underset{\sim}{v}$, $s, t \in \mathbb{R}$, respectively.

Let P and Q be the points on L_1 and L_2, respectively, such that the length of PQ is the shortest distance between the two lines.



From the vector equations, we know that $L_1 || \underset{\sim}{u}$ and $L_2 || \underset{\sim}{v}$. Then, $\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v}$ is normal to both L_1 and L_2. Hence, $\overrightarrow{PQ} || \underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v}$.

Let S_1 and S_2 be the planes that containing L_1 and L_2, respectively, such that $\overrightarrow{PQ}$ is normal to both planes. Then S_1 and S_2 are parallel planes. Hence the distance between P and Q is the distance between the two parallel planes S_1 and S_2.

Therefore, the shortest distance between L_1 and L_2

$= \left|\overrightarrow{AB} \cdot \frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}\right|$ where A and B are the points on L_1 and L_2, respectively, and, $\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v}$ is a vector normal to L_1 and L_2.

Exercise

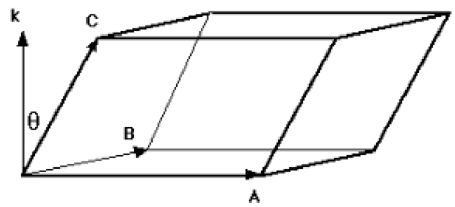
Find the shortest distance between the two lines.

$L_1: \underset{\sim}{r_1} = \underset{\sim}{a} + t\underset{\sim}{u} = \langle 0,9,2\rangle + \langle 3,-1,1\rangle t$--(a)

$L_2: \underset{\sim}{r_2} = \underset{\sim}{b} + t\underset{\sim}{v} = \langle -6,-5,10\rangle + \langle -3,2,4\rangle t$--(b)

4.1.6 Uses of Scalar Triple Products

(a)Parallelepiped



$|\mathbf{A}\times\mathbf{B}\bullet\mathbf{C}|$ = Volume of the parralelpipped

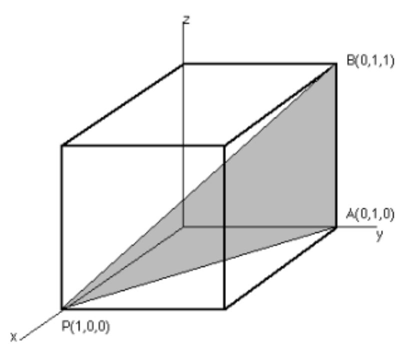
Area of Base Parralelogram = $|\mathbf{A}\times\mathbf{B}|$

$\mathbf{A}\times\mathbf{B}\bullet\mathbf{C}=|\mathbf{A}\times\mathbf{B}||\mathbf{C}|\cos\theta = BaseArea*height = \text{Volume}$

You can show that $\mathbf{A}\times\mathbf{B}\bullet\mathbf{C}=\mathbf{B}\times\mathbf{C}\bullet\mathbf{A}=\mathbf{C}\times\mathbf{A}\bullet\mathbf{B}=\mathbf{A}\bullet\mathbf{B}\times\mathbf{C}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b)Projecting area

$|\mathbf{A}\times\mathbf{B}\bullet\mathbf{n}|$=Orthogonal projection of parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ onto a plane with unit normal $\mathbf{n}$.



Find the Area of the projection of triangle PAB onto the yz plane. To do this we use the formula for the area of a projected parallelogram and half the answer to get triangular area instead.

PAxPB = $\mathbf{i}+\mathbf{j}$
Answer = .5 * $\mathbf{i}$ (dot) ($\mathbf{i}+\mathbf{j}$) = .5