ENGINEERING APPLICATIONS OF VECTOR ALGEBRA AND VECTOR ANALYSIS

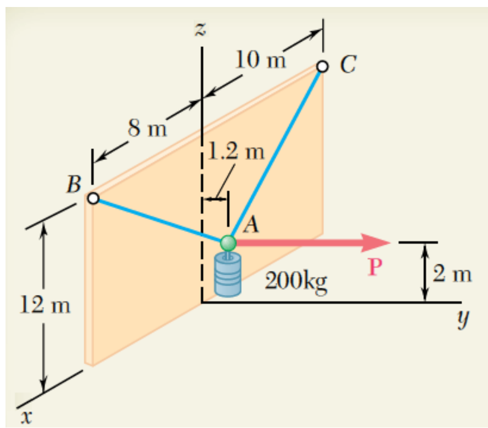
WEEK 5: ENGINEERING APPLICATIONS OF VECTOR ALGEBRA AND VECTOR ANALYSIS

5.1 ENGINEERING APPLICATION: VECTOR ALGEBRA

5.1.1 HEAD TO TAIL METHOD

Example 5.1:

1. A 200 kg cylinder is hung by means of two cables AB and AC, which are attached to the top of a vertical wall. A horizontal force ***P*** perpendicular to the wall holds the cylinder in the position shown below.



(i) Find the coordinate of position A, B and $C$and their position vectors based on the axis given in the diagram above.

(ii) Identify all the force vectors acting on point$A$. (Assume $g = 9.81ms^{-2}$)

(iii) Determine the resultant of forces acting on point A using head to-tail method.

(iv) Assume the system is in static, determine the magnitude of ***P*** and the tension in each cable.

Solution:

In this example, you should be able to solve 3D engineering problem using vector. Hint: imagine 3D object and translate it in the point point (i.e. $A = (1,2,3)$) , $B = (2,3,4)$ and vector form (i.e. position vector $\overrightarrow{OA} = \langle 1,2,3 \rangle$, $\overrightarrow{OA} = \langle 2,3,4 \rangle$ and arbitrary vector $\overrightarrow{AB} = \langle 1,1,1 \rangle$, $\overrightarrow{BA} = \langle -1,-1,-1 \rangle$)

(i) Find the coordinate of position A, B and C and their position vectors based on the axis given in the diagram.

$A = (0,1.2,2)$, $B = (8,0,12)$, $C = (-10,0,12)$

$\overrightarrow{OA} = \langle 0,1.2,2\rangle$, $\overrightarrow{OB} = \langle 8,0,12\rangle$, $\overrightarrow{OC} = \langle -10,0,12\rangle$

(ii) Identify all the force vectors acting on point A. (Assume $g = 9.81ms^{-2}$)

Horizontal force ***P*,** & its vector $\underset{\sim}{P} = \langle 0, P, 0\rangle$ (note it has magnitude P in y direction)

Vertical force 200 kg cylinder, and its vector $\underset{\sim}{W} = \langle 0,0,-1962N\rangle$ (note it has magnitude 1962N in **negative** z direction)

Tension cable AB, and its vector $\vec{T}_{AB} = \underbrace{\left|\vec{T}_{AB}\right|}_{Magnitude} \underbrace{\hat{\vec{T}}_{AB}}_{Direction\ vector}$

The magnitude is unknown (Precaution: the magnitude of vector $\left|\overrightarrow{AB}\right|$ is not equal to the magnitude of tension $|\overrightarrow{T_{AB}}|$)

The direction vector (unit vector) is the same as the unit vector of $\hat{\overrightarrow{AB}}$

$$\vec{AB} = \vec{OB} - \vec{OA} = \langle 8,0,12\rangle - \langle 0,1.2,2\rangle = \langle 8,-1.2,10\rangle$$

$$\hat{\vec{AB}} = \frac{\langle 8,-1.2,10\rangle}{\sqrt{8^2 + (-1.2)^2 + 10^2}} = \frac{\langle 8,-1.2,10\rangle}{\sqrt{165.44}} = \langle 0.6220,-0.09330,0.7775\rangle$$

$$\vec{T}_{AB} = \underbrace{\left|\vec{T}_{AB}\right|}_{Magnitude} \underbrace{\hat{\vec{T}}_{AB}}_{Direction\ vector} = \left|\vec{T}_{AB}\right| \hat{\vec{AB}} = \left|\vec{T}_{AB}\right| \langle 0.6220,-0.09330,0.7775\rangle$$

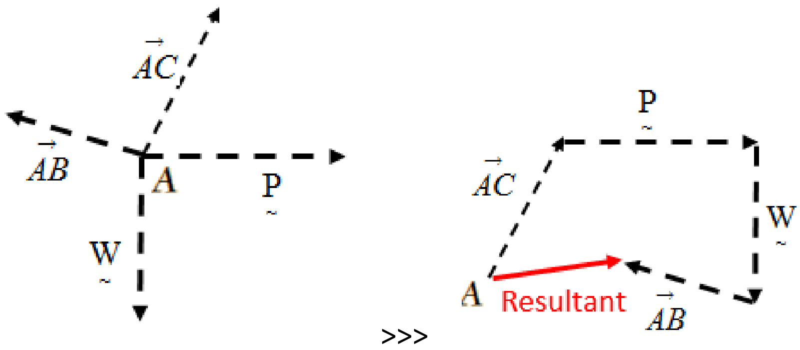
Tension cable AC, & its vector $\vec{T}_{AC} = \underbrace{\left|\vec{T}_{AC}\right|}_{Magnitude} \underbrace{\hat{\vec{T}}_{AC}}_{Direction\ vector}$

$$\vec{AC} = \vec{OC} - \vec{OA} = \langle -10,0,12\rangle - \langle 0,1.2,2\rangle = \langle -10,-1.2,10\rangle$$

$$\hat{\vec{AC}} = \frac{\langle -10,-1.2,10\rangle}{\sqrt{(-10)^2 + (-1.2)^2 + 10^2}} = \frac{\langle -10,-1.2,10\rangle}{\sqrt{201.44}} = \langle -0.7046,-0.08455,0.7046\rangle$$

$$\vec{T}_{AC} = \underbrace{\left|\vec{T}_{AC}\right|}_{Magnitude} \underbrace{\hat{\vec{T}}_{AC}}_{Direction\ vector} = \left|\vec{T}_{AC}\right| \hat{\vec{AC}} = \left|\vec{T}_{AC}\right| \langle -0.7046,-0.08455,0.7046\rangle$$

(iii) Determine the resultant of forces acting on point A using head to-tail method.



The resultant of force is the addition of vectors $\vec{AB}$, $\vec{AC}$, $\underset{\sim}{\mathrm{P}}$, and $\underset{\sim}{\mathrm{W}}$ where

Resultant, ΣF

$$=\left|\vec{T}_{AC}\right|\langle -0.7046, -0.08455, 0.7046\rangle + \left|\vec{T}_{AB}\right|\langle 0.6220, -0.09330, 0.7775\rangle + \langle 0, P, 0\rangle + \langle 0, 0, -1962\text{N}\rangle$$

$$=\Big\langle \{-0.7046\left|\vec{T}_{AC}\right| + 0.6220\left|\vec{T}_{AB}\right|\}, \{-0.08455\left|\vec{T}_{AC}\right| - 0.09330\left|\vec{T}_{AB}\right| + P\}, \{0.7046\left|\vec{T}_{AC}\right| + 0.7775\left|\vec{T}_{AB}\right| -1962\text{N}$$

(iv) **Assume the system is in static, determine the magnitude of P and the tension in each cable.**

Since the object is in equilibrium, hence the resultant of force at point A must be equal to zero, $\Sigma F = 0$

$$\Big\langle \{-0.7046\left|\vec{T}_{AC}\right| + 0.6220\left|\vec{T}_{AB}\right|\}, \{-0.08455\left|\vec{T}_{AC}\right| - 0.09330\left|\vec{T}_{AB}\right| + P\}, \{0.7046\left|\vec{T}_{AC}\right| + 0.7775\left|\vec{T}_{AB}\right| -1962\text{N}\}\Big\rangle = \underset{\sim}{0}$$

$$\Sigma F_x = 0$$

$$\{-0.7046\left|\vec{T}_{AC}\right| + 0.6220\left|\vec{T}_{AB}\right|\} = \underset{\sim}{0} \quad >>> \quad \left|\vec{T}_{AC}\right| = \frac{0.6220}{0.7046}\left|\vec{T}_{AB}\right|$$

$$\Sigma F_z = 0$$

$$0.7046\left(\frac{0.6220}{0.7046}\left|\vec{T}_{AB}\right|\right)+0.7775\left|\vec{T}_{AB}\right|-1962\text{N}=0 \;>>>\; \left|\vec{T}_{AB}\right|=1401.93\text{N}$$

$$\left|\vec{T}_{AC}\right|=\frac{0.6220}{0.7046}(1401.93)=1237.58\text{N}$$

$$\Sigma F_y=0$$

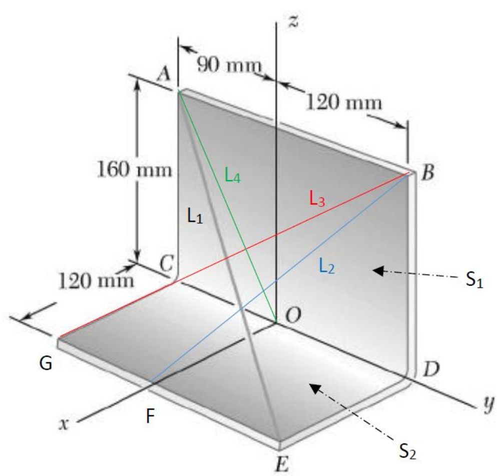
$$-0.08455\left|\vec{T}_{AC}\right|-0.09330\left|\vec{T}_{AB}\right|+P=0 \;>>>$$

$$P=235.44\text{N}$$

5.1.2 ENGINEERING APPLICATION: VECTOR IN 3 D SPACE

Example 5.2:

The wire AE, L_1 is stretched between the corners A and E of a bent plate. The wire BF, L_2 is stretched between the position B and F. The wire BG, L_3 is stretched between the position B and G. The wire OA, L_4 is stretched between the position O and A.



(i) Find the vector equation of line for wire AE, BF, BG and OA. Hence find the intersection point between line L_1 with L_2 ; line L_3 with L_4separately if exist.

(ii) Find the equation of plane for S_1 from point 0, A & B and equation of plane for S_2 from point E, F and G if possible. Hence find the intersection line between S_1 and S_2 if exist.

(iii) Given the intersection point between L_1 & L_3 is $\boldsymbol{P}(\mathbf{60, 15, 80})$. Find the shortest distance between intersection point L_1 & L_3 and plane S_1 and plane S_2 respectively.

Solution:

(i)

Vector equation of line for wire BF

Point $B=(0,120,160)$ and $F=(120,0,0)$

Position vector, $\vec{OB}=\langle 0,120,160\rangle$, $\vec{OF}=\langle 120,0,0\rangle$

Vector $\vec{BF}=\vec{OF}-\vec{OB}=\langle 120,0,0\rangle-\langle 0,120,160\rangle=\langle 120,-120,-160\rangle$

$\underset{\sim}{L_2}=\vec{OB}+t\vec{BF}=\langle 0,120,160\rangle+t\langle 120,-120,-160\rangle$

Vector equation of line for wire BG

Point $B=(0,120,160)$ and $G=(120,-90,0)$

Position vector, $\vec{OB}=\langle 0,120,160\rangle$, $\vec{OG}=\langle 120,-90,0\rangle$

Vector $\vec{BG}=\vec{OG}-\vec{OB}=\langle 120,-90,0\rangle-\langle 0,120,160\rangle=\langle 120,-210,-160\rangle$

$\underset{\sim}{L_3}=\vec{OB}+u\vec{BG}=\langle 0,120,160\rangle+u\langle 120,-210,-160\rangle$

Vector equation of line for wire AE

Point $A=(0,-90,160)$ and $E=(120,120,0)$

Position vector, $\vec{OA}=\langle 0,-90,160\rangle$, $\vec{OE}=\langle 120,120,0\rangle$

Vector $\vec{AE}=\vec{OE}-\vec{OA}=\langle 120,120,0\rangle-\langle 0,-90,160\rangle=\langle 120,210,-160\rangle$

$\underset{\sim}{L_1}=\vec{OA}+s\vec{AE}=\langle 0,-90,160\rangle+s\langle 120,210,-160\rangle$

Vector equation of line for wire OA

Point $O=(0,0,0)$ and $A=(0,-90,160)$

Position vector, $\vec{OO}=\langle 0,0,0\rangle$, $\vec{OA}=\langle 0,-90,160\rangle$

Vector $\vec{OA}=\vec{OA}-\vec{OO}=\langle 0,-90,160\rangle-\langle 0,0,0\rangle=\langle 0,-90,160\rangle$

$\underset{\sim}{L_4}=\vec{OO}+v\vec{OA}=\langle 0,0,0\rangle+v\langle 0,-90,160\rangle=v\langle 0,-90,160\rangle$

To check existence of intersection point between L1 & L2:

$\underset{\sim}{L_1}=\vec{OA}+s\vec{AE}=\langle 0,-90,160\rangle+s\langle 120,210,-160\rangle$

$\underset{\sim}{L_2}=\vec{OB}+s\vec{BF}=\langle 0,120,160\rangle+t\langle 120,-120,-160\rangle$

$\underset{\sim}{L_1}=\underset{\sim}{L_2}$

$\langle 0,-90,160\rangle+s\langle 120,210,-160\rangle=\langle 0,120,160\rangle+t\langle 120,-120,-160\rangle$

$120s=120t$ >>> $s=t$ --(1)

$160-160s=160-160t$ >>> $s=t$ --(2)

$-90+210s=120-120t$ >> $\frac{-210}{-330}=t$ --(3)

$LHS: -90+210(\frac{-210}{-330})=43.6364$

$RHS: 120-120(\frac{-210}{-330})=43.6364$

$LHS=RHS$

There is intersection point between L1 & L2.

$$\underset{\sim}{L_1} = \underset{\sim}{L_2}$$

$$= \langle 0, -90, 160 \rangle + \frac{210}{330} \langle 120, 210, -160 \rangle$$

$$= \langle 0, 120, 160 \rangle + \frac{210}{330} \langle 120, -120, -160 \rangle$$

$$= \langle 76.3636 \quad 43.6364 \quad 58.1818 \rangle$$

The intersection point is at $(76.3636, 43.6364, 58.1818)$

To check existence of intersection point between L3 & L4:

$$\underset{\sim}{L_3} = \vec{OB} + u\vec{BG} = \langle 0, 120, 160 \rangle + u \langle 120, -210, -160 \rangle$$

$$\underset{\sim}{L_4} = \vec{OO} + v\vec{OA} = \langle 0, 0, 0 \rangle + v \langle 0, -90, 120 \rangle = v \langle 0, -90, 120 \rangle$$

$$\underset{\sim}{L_3} = \underset{\sim}{L_4}$$

$$\langle 0, 120, 160 \rangle + u \langle 120, -210, -160 \rangle = v \langle 0, -90, 160 \rangle$$

$$120u = 0 \;>>>\; u = 0$$

$$120 - 210u = -90v \;>>>\; v = -\tfrac{4}{3}$$

$$160 - 160u = 160v$$

$$LHS: 160 - 160(0) = 160$$

$$RHS: 160(-\tfrac{4}{3}) = -213.33$$

$$LHS \neq RHS$$

There is no intersection point between L3 & L4.

(ii)

To form equation of plane for S_1, we need three points located on the plane:

Point $A = (0, -90, 160)$

Point $B = (0, 120, 160)$

Point $O = (0, 0, 0)$

The format for equation of plane: $ax + by + cz = k$ where the normal vector to the plane is $\langle a, b, c \rangle$

Normal vector to the plane = $\vec{OA} x \vec{OB} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & -90 & 160 \\ 0 & 120 & 160 \end{vmatrix} = \langle -33600, 0, 0 \rangle$

Hence we get, $ax + by + cz = k$ >>> $-33600x + 0y + 0z = k$ >>> $k = -33600x$

From the point located on S_1, i.e., point O, A and B. All the component at x direction is 0. Sub to the equation we get $k = -33600x = 0$; Thus, the plane of equation is $-33600x = 0$ which can be simplify to $x = 0$

Note: normal vector $\langle -33600, 0, 0 \rangle$ is parallel to $\langle 1, 0, 0 \rangle$

To form equation of plane for S_2, we need three points located on the plane:

Point $E = (120, 120, 0)$

Point $F = (120, 0, 0)$

Point $G = (120, -90, 0)$

The format for equation of plane: $ax + by + cz = k$ where the normal vector to the plane is $\langle a, b, c \rangle$

Normal vector to the plane = $\vec{EF} x \vec{EG} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & -120 & 0 \\ 0 & -210 & 0 \end{vmatrix} = \langle 0,0,0 \rangle$

The normal vector is a zero vector, this result is invalid because the points that we selected to form a plane must be non-parallel. Note that we can't find the normal vector by using cross product of parallel vectors as $\underset{\sim}{v_1} x \underset{\sim}{v_2} = |\underset{\sim}{v_1}||\underset{\sim}{v_2}| \sin 0 \ \underset{\sim}{n} = \underset{\sim}{0}$ However, in this case, the vector $\vec{EF}$ and $\vec{EG}$ are in parallel. Therefore, it is impossible to use the selected points to calculate the plane of equation.

Thus a new point $O = (0,0,0)$ is selected to avoid this issue.

Normal vector to the plane = $\vec{OE} x \vec{OF} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 120 & 120 & 0 \\ 120 & 0 & 0 \end{vmatrix} = \langle 0,0,-14400 \rangle$

Hence we get, $ax + by + cz = k$ >>> $0x + 0y - 14400z = k$ >>> $k = -14400z$

From the point located on S_2, i.e., point O, E and F. All the component at z direction is 0. Sub to the equation we get $k = -14400z = 0$; Thus, the plane of equation is $-14400z = 0$ which can be simplify to $z = 0$

Note: normal vector $\langle 0,0,-14400 \rangle$ is parallel to $\langle 0,0,1 \rangle$

(iii)

Equation of plane for S_1 is x=0 while S_2 is z=0. The normal vector to plane S_1 is $\langle 1,0,0\rangle$ and normal vector to plane S_2 is $\langle 0,0,1\rangle$

Thus, the arbitrary point at plane S_2 is $(\mathrm{x\ y\ 0})$ while arbitrary point at plane S_1 is $(0\ \mathrm{y\ z})$

Let point M = $(1\ 1\ 0)$ located on plane S_2 while point N = $(0\ 1\ 1)$ located on plane S_1

$$\vec{MP} = \vec{OP} - \vec{OM} = \langle 60\ 15\ 80\rangle - \langle 1\quad 1\quad 0\rangle = \langle 59\ 14\ 80\rangle$$

Shortest distance between intersection point L_1 & L_3 and plane S_2 is

$$\left|\vec{MP}.\langle 0\ 0\ 1\rangle\right| = \left|\langle 59\ 14\ 80\rangle.\langle 0\ 0\ 1\rangle\right| = 80unit$$

$$\vec{NP} = \vec{OP} - \vec{ON} = \langle 60\ 15\ 80\rangle - \langle 0\quad 1\quad 1\rangle = \langle 60\ 14\ 79\rangle$$

Shortest distance between intersection point L_1 & L_3 and plane S_1 is

$$\left|\vec{NP}.\langle 1\ 0\ 0\rangle\right| = \left|\langle 60\ 14\ 79\rangle.\langle 1\ 0\ 0\rangle\right| = 60unit$$

5.2 ENGINEERING APPLICATION: VECTOR ANALYSIS

5.2.1 NAVIGATION

The word problems encountered most often with vectors are navigation problems. These navigation problems use variables like speed and direction to form vectors for computation. Some navigation problems ask us to find the groundspeed of an aircraft using the combined forces of the wind and the aircraft. For these problems it is important to understand the resultant of two forces and the components of force.

Each of the three vectors in the triangle of velocities has two properties – magnitude and direction. This means that there are a total of six components. These are the True Air Speed (TAS) and heading (HDG) of the aircraft, the speed and direction of the wind (W/V), and the Ground Speed (GS) and track (TR) of the path over the ground. This is shown in Figure 5.1.

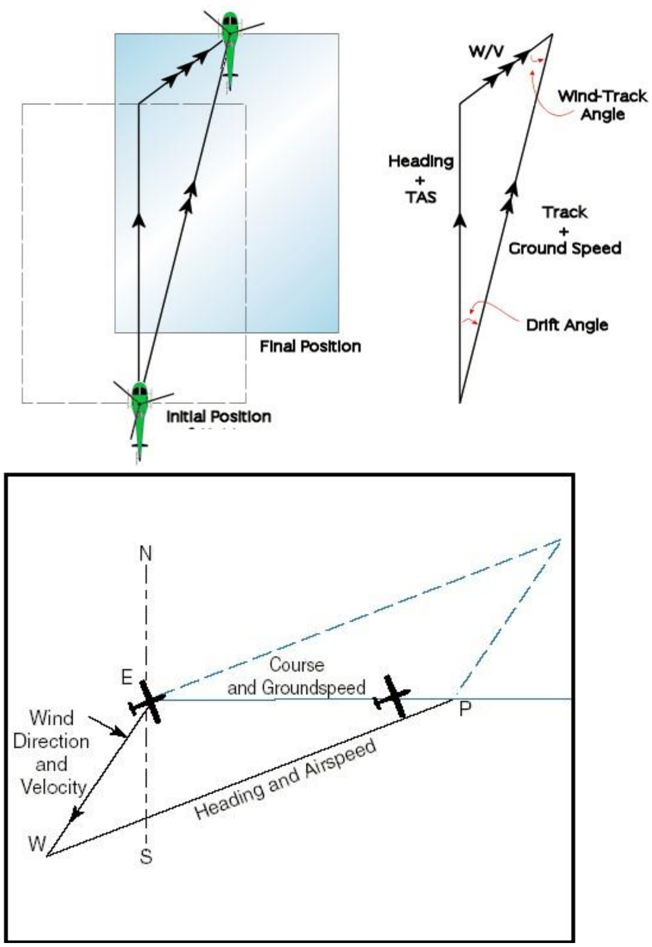


Figure 5.1: Triangle of Vectors (Velocities)

To summarize:

Course—the direction of a line drawn on a chart representing the intended airplane path, expressed as the angle measured from a specific reference datum clockwise from 0° through 360° to the line.

Heading—is the direction in which the nose of the airplane points during flight.

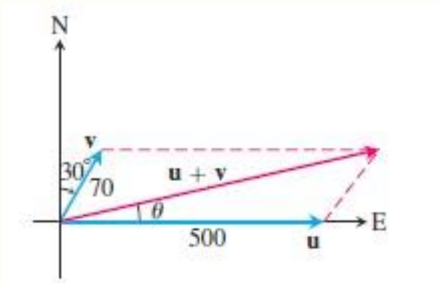
Drift angle—is the angle between heading and track.

Airspeed—is the rate of the airplane's progress through the air.

Groundspeed—is the rate of the airplane's in-flight progress over the ground.

Example 5.3

A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?



Solution If $\mathbf{u}$ = the velocity of the airplane alone and $\mathbf{v}$ = the velocity of the tailwind, then $|\mathbf{u}| = 500$ and $|\mathbf{v}| = 70$ (Figure 12.17). The velocity of the airplane with respect to the ground is given by the magnitude and direction of the resultant vector $\mathbf{u} + \mathbf{v}$. If we let the positive x-axis represent east and the positive y-axis represent north, then the component forms of $\mathbf{u}$ and $\mathbf{v}$ are

$$\mathbf{u} = \langle 500, 0 \rangle \qquad \text{and} \qquad \mathbf{v} = \langle 70\cos 60°, 70\sin 60° \rangle = \langle 35, 35\sqrt{3} \rangle.$$

Therefore,

$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle = 535\mathbf{i} + 35\sqrt{3}\,\mathbf{j}$$

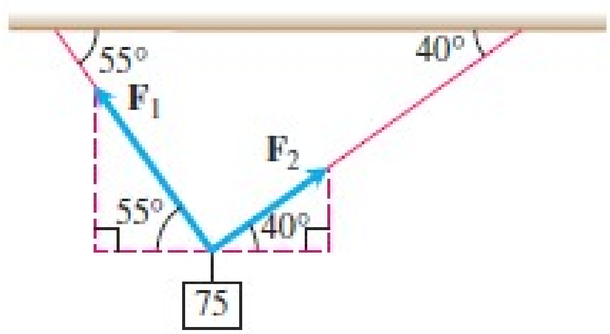
$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

and

$$\theta = \tan^{-1}\frac{35\sqrt{3}}{535} \approx 6.5°.$$

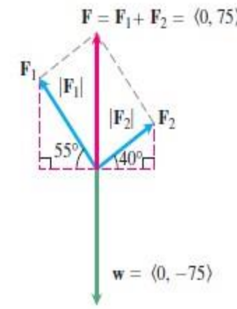
The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east. ■

Example 5.4



A 75 N weight is suspended by two wires as shown in figure above. Find the forces F_1 and F_2 acting in both wires.

Solution The force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ have magnitudes $|\mathbf{F}_1|$ and $|\mathbf{F}_2|$ and components that are measured in Newtons. The resultant force is the sum $\mathbf{F}_1 + \mathbf{F}_2$ and must be equal in magnitude and acting in the opposite (or upward) direction to the weight vector $\mathbf{w}$



$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°, |\mathbf{F}_1|\sin 55°\rangle \quad \text{and} \quad \mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°, |\mathbf{F}_2|\sin 40°\rangle.$$

Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75\rangle$, the resultant vector leads to the system of equations

$$-|\mathbf{F}_1|\cos 55° + |\mathbf{F}_2|\cos 40° = 0$$
$$|\mathbf{F}_1|\sin 55° + |\mathbf{F}_2|\sin 40° = 75.$$

Solving for $|\mathbf{F}_2|$ in the first equation and substituting the result into the second equation, we get

$$|\mathbf{F}_2| = \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°} \quad \text{and} \quad |\mathbf{F}_1|\sin 55° + \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°}\sin 40° = 75.$$

It follows that

$$|\mathbf{F}_1| = \frac{75}{\sin 55° + \cos 55° \tan 40°} \approx 57.67 \text{ N},$$

and

$$|\mathbf{F}_2| = \frac{75 \cos 55°}{\sin 55° \cos 40° + \cos 55° \sin 40°}$$
$$= \frac{75 \cos 55°}{\sin(55° + 40°)} \approx 43.18 \text{ N}.$$

The force vectors are then $\mathbf{F}_1 = \langle -33.08, 47.24\rangle$ and $\mathbf{F}_2 = \langle 33.08, 27.76\rangle$.

Exercises

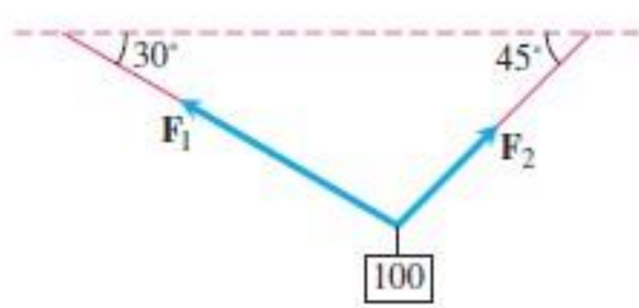
1) A boat leaves port on a heading of 40° with the automatic pilot set for 12 knots. On this particular day, there is a 6-knot current with a heading of 75°.

 a. Sketch and label vectors to represent the intended path of the boat, the current and the resultant path of the boat with the effects of the current.

 b. Calculate the speed and heading at which the boat will actually travel due to the effects of the current.

2) A plane leaves the airport on a heading 45° traveling a 400 mph. The wind is blowing at a heading of 135° at a speed of 40 mph. What is the actual velocity of the plane?

3) Consider a 100-N weight suspended by two wires as shown in the accompanying figure. Find the magnitudes and components of the force vectors **F**1 and **F**2.



5.2.2 DOT PRODUCT (WORK DONE)

After investigating the dot product, we apply it to finding the projection of one vector onto another (as displayed in Figure 5.2) and to finding the work done by a constant force acting through a displacement. The scalar quantity we seek is the length $|F|\cos\theta$ where θ is the angle between the two vectors **F** and D. Then

$$\begin{aligned} \text{Work} &= \begin{pmatrix}\text{scalar component of } \mathbf{F} \\ \text{in the direction of } \mathbf{D}\end{pmatrix}(\text{length of } \mathbf{D}) \\ &= (|\mathbf{F}|\cos\theta)|\mathbf{D}| \\ &= \mathbf{F}\cdot\mathbf{D}. \end{aligned}$$

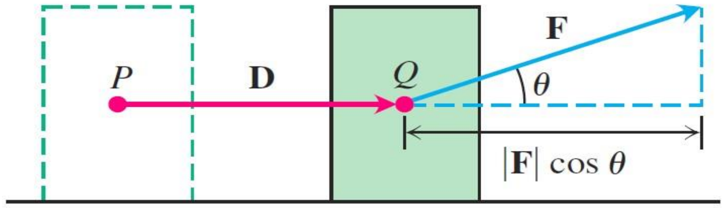


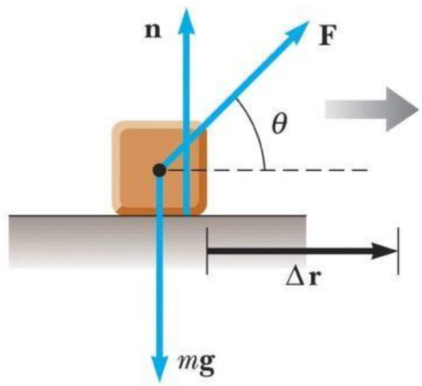
Figure 5.2: The work done by a constant force **F** during a displacement **D i**s $|F|\cos\theta$, which is the dot product **F.D**.

DEFINITION The **work** done by a constant force **F** acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D}.$$

Forces Perpendicular to the Motion Do No Work

When an object is displaced horizontally on a flat table, the normal force n and the gravitational force Fg do no work since cos θ = 90◦ = 0



<u>Example 5.5</u>

If $|F|$ = 40 N (newtons), D = $|D|$ = 3 m, and θ = 60^o, the work done by Fin acting from P to Q is

$$\begin{aligned} \text{Work} &= \mathbf{F} \cdot \mathbf{D} && \text{Definition} \\ &= |\mathbf{F}||\mathbf{D}|\cos\theta \\ &= (40)(3)\cos 60^\circ && \text{Given values} \\ &= (120)(1/2) = 60 \text{ J (joules)}. \end{aligned}$$

Exercises

1) How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?

2) A 30 kg box is placed 10 m up a ramp that is inclined at 23° to the horizontal. Calculate the work done by the force of gravity as the box slides down to the bottom of the ramp.

5.2.3 TORQUE (CROSS PRODUCT)

When we turn a bolt by applying a force **F** to a wrench (Figure 5.3), we produce a torque that causes the bolt to rotate.

The **torque vector** points in the direction of the axis of the bolt according to the right-hand rule (so the rotation is counter clockwise when viewed from the *tip* of the vector).

The magnitude of the torque depends on how far out on the wrench the force is applied and on how much of the force is perpendicular to the wrench at the point of application.

The number we use to measure the torque's magnitude is the product of the length of the lever arm **r** and the scalar component of **F** perpendicular to **r**.

$$Magnitude\ of\ torque\ vector = |r||F|sin\theta,\ or \quad |r \times F|$$

If we let **n** be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is

$$Torque\ vector = (|r||F|sin\theta)n, \qquad or \quad r \times F$$

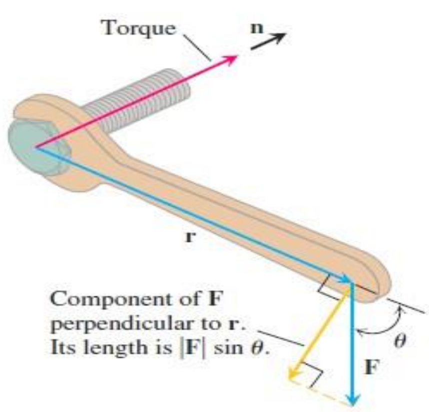


Fig 5.3: The torque vector describes the tendency of the force **F** to drive the bolt forward.

<u>Example 5.6</u>

Find the magnitude of the torque generated by force ***F*** at the pivot point ***P*** in Figure 5.4 is

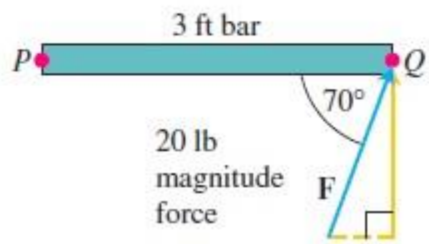


Figure 5.4: Torque exerted by **F** at ***P***

$$\begin{aligned}|\overrightarrow{PQ} \times \mathbf{F}| &= |\overrightarrow{PQ}||\mathbf{F}| \sin 70^\circ \\ &\approx (3)(20)(0.94) \\ &\approx 56.4 \text{ ft-lb}.\end{aligned}$$

The magnitude of the torque exerted by **F** at *P* is about 56.4 ft-lb. The bar rotates counterclockwise around *P*.

5.2.4 TRIPLE SCALAR OR BOX PRODUCT

The product is called the **triple scalar product** of **u**, **v**, and **w** (in that order). As you can see from the formula

$$|(u \times v) \cdot w| = |u \times v||w||cos\theta|$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by **u**, **v**, and **w** (Figure 5.5). The number $|(u \times v)|$ is the area of the base parallelogram. The number $|w||cos\theta|$is the parallelepiped's height. Because of this geometry, $(u \times v) \cdot w$ is also called the **box product** of **u**, **v**, and **w**.

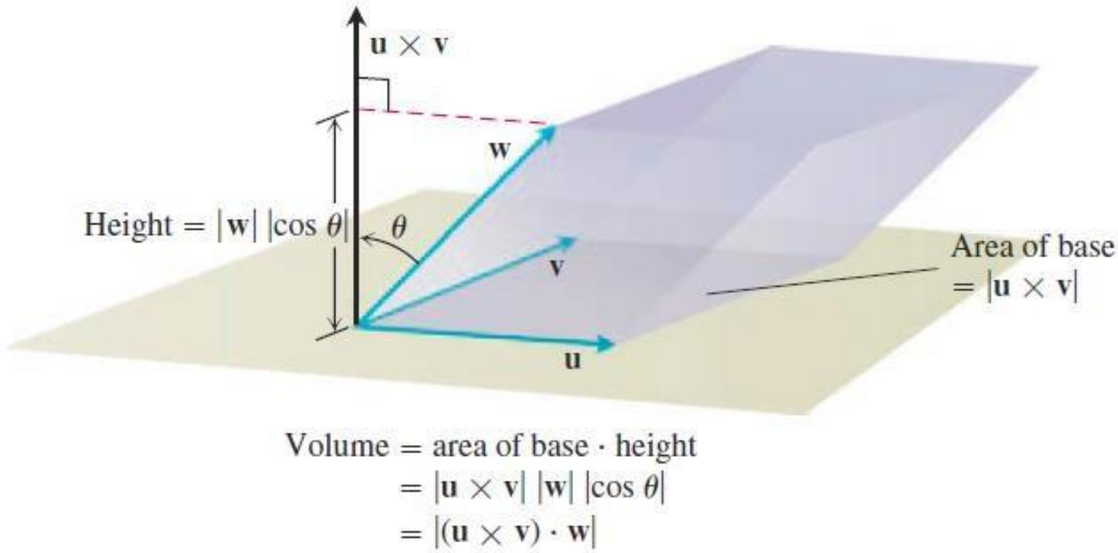


Figure 5.5: The number $|(u \times v) \cdot w| =$ is the volume of a parallelepiped.

By treating the planes of **v** and **w** and of **w** and **u** as the base planes of the parallelepiped determined by **u**, **v**, and **w**, we see that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

Since the dot product is commutative, we also have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right] \cdot \mathbf{w} \\
&= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
\end{aligned}$$

Example 5.7

Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating determinants, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.

Exercises

Find the volume of the parallelepiped (box) determined by **u**, **v**, and **w**.

	u	v	w
1)	$2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
2)	$\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
3)	$2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$

5.2.5 INTERPRETATION OF THE DIRECTIONAL DERIVATIVES & GRADIENT

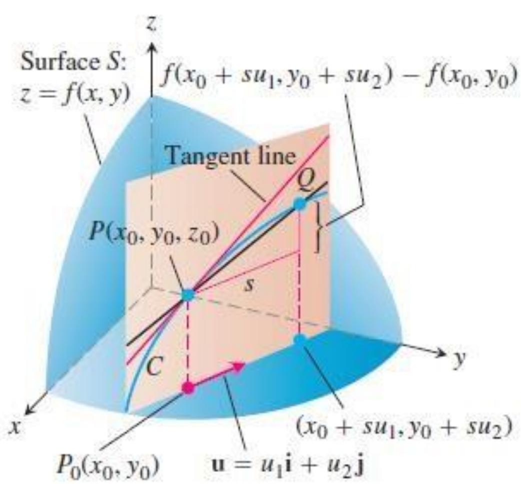


Figure 5.6: Interpretation of Directional Derivatives.

From the Figure 5.6, the slope of curve ***C*** at ***Po*** is ***(Duf)$_{Po}$*** in which it generalizes two partial derivatives. We can now ask for the rate of change of *f* in any direction **u**, not just the directions **i** and **j**.

For a physical interpretation of the directional derivative, suppose that ***T = f (x, y)*** is the temperature at each point **(*x*, *y*)** over a region in the plane. Then ***f*** (x_o, y_o) is the temperature at the point P_o (x_o, y_o) and $(D_uf)_{P_o}$ is the instantaneous rate of change of the temperature at ***Po*** stepping off in the direction **u**.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Example 5.8

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy|_{(1,1,0)} = -2, \qquad f_z = -1|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

Example 5.9

Suppose that the temperature T at each point (x, y, z) in a region of space is given by

$$T = 100 - x^2 - y^2 - z^2,$$

and that $\mathbf{F}(x, y, z)$ is defined to be the gradient of T. Find the vector field $\mathbf{F}$.

Solution The gradient field $\mathbf{F}$ is the field $\mathbf{F} = \nabla T = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$. At each point in space, the vector field $\mathbf{F}$ gives the direction for which the increase in temperature is greatest.

Exercises

Find the directions in which the functions increase and decrease most rapidly at P_o. Then find the derivatives of the functions in these directions.

(a) $f(x, y, z) = \ln xy + \ln yz + \ln xz$, $P_o(1, 1, 1)$

(b) $g(x, y, z) = xe^y + z^2$, $P_o(1, \ln 2, 1/2)$

(c) $f(x, y, z) = (x/y) - yz$, $P_o(4,1,1)$

5.2.6 GRADIENTS,TANGENTS AND NORMAL TO LEVEL CURVES

The streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach the ocean as quickly as possible. Therefore, the fastest instantaneous rate of change in a stream's elevation above sea level has a particular direction. In this section, you will see why this direction, called the "downhill" direction, is perpendicular to the contours.

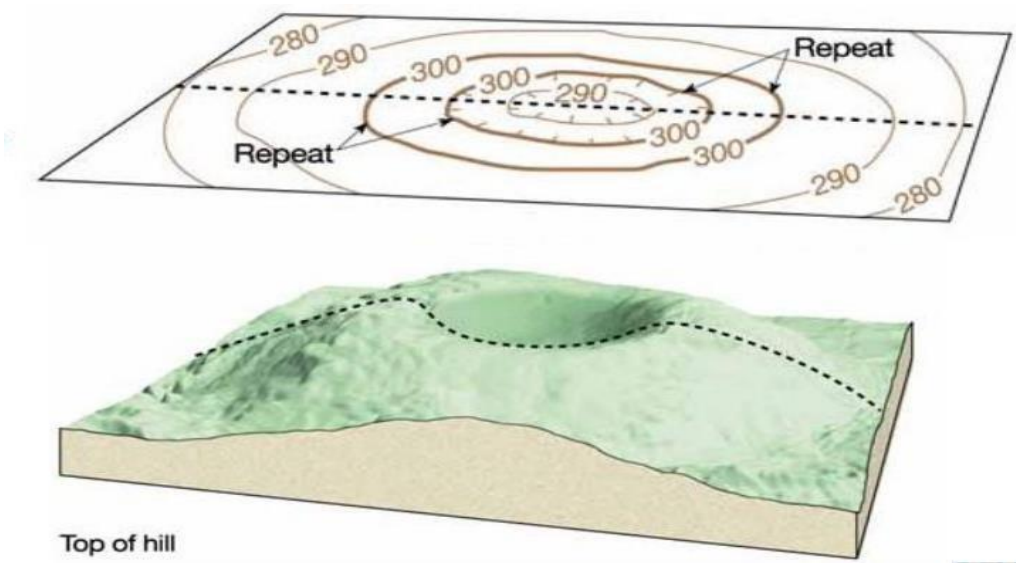


Figure 5.7: Contours of the hill

Based on above contours:

> At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure).

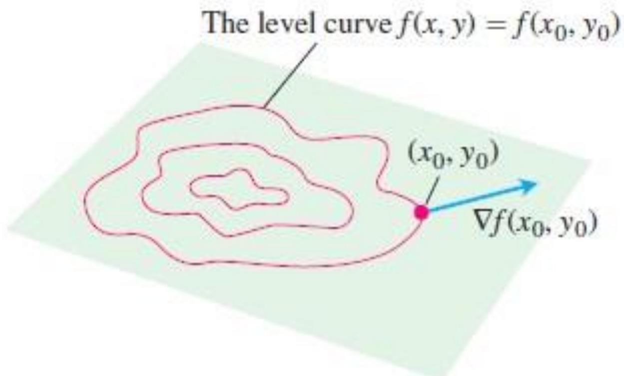


Figure 5.8: The gradient of a differentiable function of two variables at a point is always normal to the function's level curve through that point.

Where it shows our observation that streams flow perpendicular to the contours in topographical maps (see Figure 5.7). Since the downflowing stream will reach its destination in the fastest way, it must flow in the direction of the negative gradient vectors from Property 2 for the directional derivative.

Now let us restrict our attention to the curves that pass through P_o (Figure Below). All the velocity vectors at P_o are orthogonal to ∇f at P_o so the curves' tangent lines all lie in the plane through P_o normal to ∇f. At every point along the curve, ∇f is orthogonal to the curve's velocity vector.

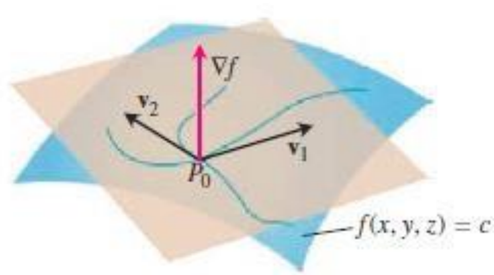


Figure Above: The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_o. The velocity vectors at P_o therefore lie in a common plane, which we call the tangent plane at P_o.

We now define this plane.

DEFINITIONS The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

The tangent plane and normal line have the following equations:

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \tag{2}$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \tag{3}$$

Example 5.10

Find an equation for the tangent to the ellipse at the point $(-2, 1)$.

$$\frac{x^2}{4} + y^2 = 2$$

Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

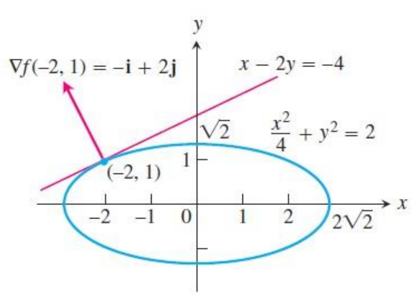
The gradient of f at $(-2, 1)$ is

$$\nabla f|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

The tangent is the line

$$(-1)(x + 2) + (2)(y - 1) = 0$$

$$x - 2y = -4.$$



We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$

Example 5.11

Find the tangent plane and normal line of the surface

$f(x, y, z) = x^2 + y^2 + z - 9 = 0$ A circular paraboloid

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Figure

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

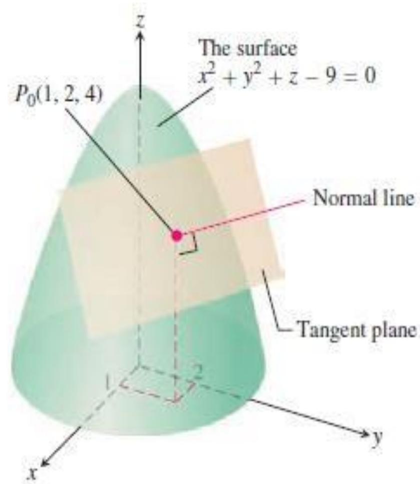
$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$



Exercises

Find equations for the

(a) Tangent plane and **(b)** normal line at the point P_o on the given surface.

5.2.7 APPLICATION OF DIVERGENCE AND CURL

Divergence at a given point measures the net flow out of a small box around the point, that is, it measures what is produced (source) or consumed (sink) at a given point in space.

For example, it is used to describe the flow of gas within a domain space. A gas is compressible, unlike a liquid, and the divergence of its velocity field measures to what extent it is expanding or compressing at each point. Intuitively, if a gas is expanding at the point (x_o, y_o) the lines of flow would diverge there (hence the name) and, since the gas would be flowing out of a small rectangle about (x_o, y_o), the divergence of **F** at (x_o, y_o) would be positive. If the gas were compressing instead of expanding, the divergence would be negative.

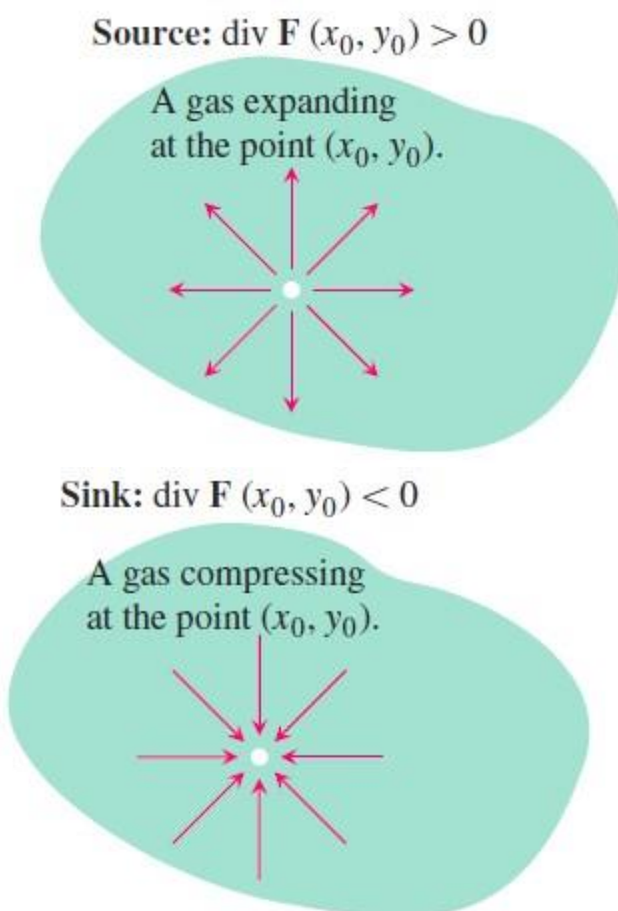


Figure 5.9: If a gas is expanding at a point the lines of flow have positive divergence; if the gas is compressing, the divergence is negative. Otherwise, it will get zero.

Example 5.12

Determine the divergence's characteristic of the vector fields; $F(x, y) = 3x^2\boldsymbol{i} - 6xy\boldsymbol{j}$

$$\begin{aligned}\nabla \cdot \boldsymbol{F}(x,y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}3x^2 + \frac{\partial}{\partial y}(-6xy) = 6x - 6x = 0\,.\end{aligned}$$

A vector field with vanishing divergence is called a solenoidal vector field.

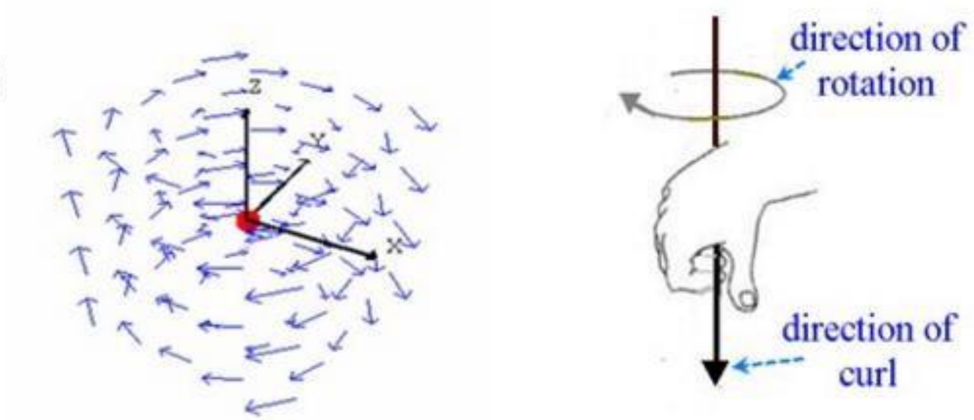
*In fluid dynamics, when the velocity field of a flowing liquid always has divergence equal to zero, as in those cases, the liquid is said to be **incompressible**.

If we think of the vector field as a velocity vector field of a fluid in a motion, the curl measures the rotation. At a given point, the curl is a vector parallel to the axis of rotation of flow lines near the point, with direction determined by the Right Hand Rule.

Example 5.13

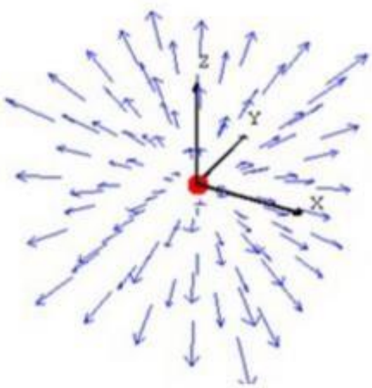
Determine wherther the curl of each vector field at the origin is the zero vector or points in the certain directions as $\pm i$, $\pm j$, $\pm k$.

(a)



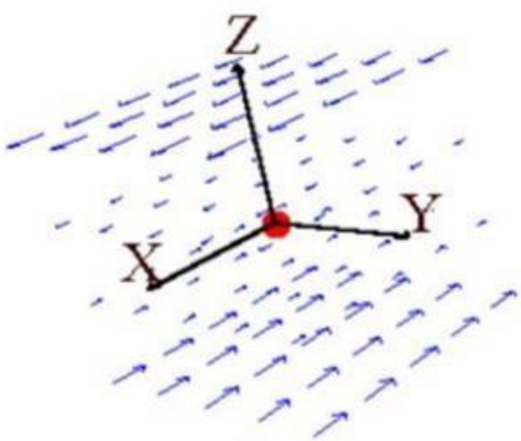
Based on the direction of the rotation and the Right Hand Rule, the curl will point in the $-\mathbf{k} = \langle 0,0,-1 \rangle$ direction.

(b)



The vector field is clearly *irrotational*, thus the curl is the zero vector: $\langle 0,0,0\rangle$.

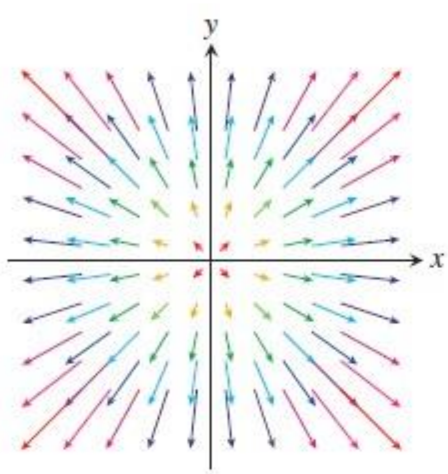
(c)



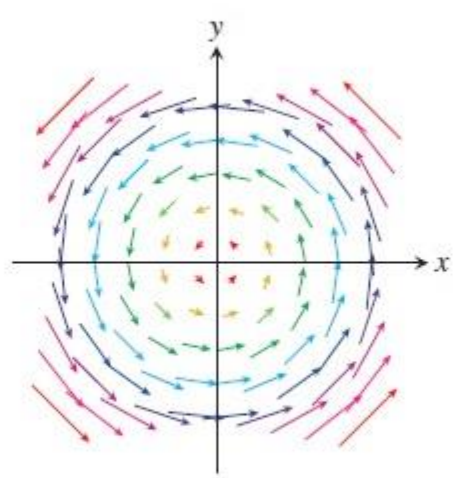
Based on the direction of the rotation and the Right Hand Rule, the curl points in the direction of
$+\mathbf{j} = \langle 0,1,0\rangle$

Example 5.14

The following vector fields represent the velocity of a gas flowing in the *xy*-plane. Find the divergence and curl of each vector field and interpret its physical meaning. Figure displays the vector fields.



(a)



(b)

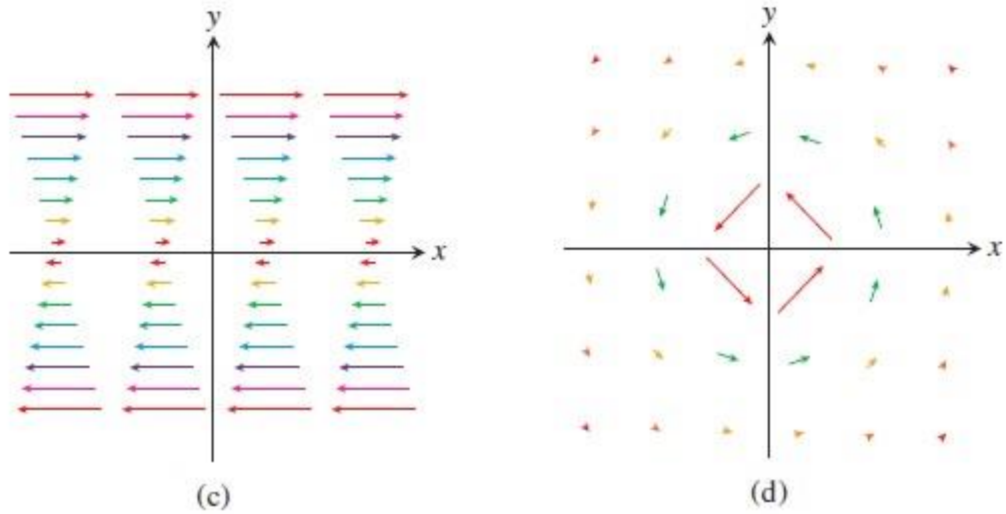


Figure 5.10: Velocity fields of a gas flowing in the plane

DIVERGENCE

Solution

(a) div $\mathbf{F} = \frac{\partial}{\partial x}(cx) + \frac{\partial}{\partial y}(cy) = 2c$: If $c > 0$, the gas is undergoing uniform expansion; if $c < 0$, it is undergoing uniform compression.

(b) div $\mathbf{F} = \frac{\partial}{\partial x}(-cy) + \frac{\partial}{\partial y}(cx) = 0$: The gas is neither expanding nor compressing.

(c) div $\mathbf{F} = \frac{\partial}{\partial x}(y) = 0$: The gas is neither expanding nor compressing.

(d) div $\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0$: Again, the divergence is zero at all points in the domain of the velocity field.

CURL

Solution

(a) *Uniform expansion:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx) = 0$. The gas is not circulating at very small scales.

(b) *Rotation:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\sigma}{\partial x}(cx) - \frac{\sigma}{\partial y}(-cy) = 2c$. The constant circulation density indicates rotation at every point. If $c > 0$, the rotation is counterclockwise; if $c < 0$, the rotation is clockwise.

(c) *Shear:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = -\frac{\partial}{\partial y}(y) = -1$. The circulation density is constant and negative, so a paddle wheel floating in water undergoing such a shearing flow spins clockwise. The rate of rotation is the same at each point. The average effect of the fluid flow is to push fluid clockwise around each of the small circles shown in Figure 16.31.

(d) *Whirlpool:*

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0.$$

The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined.