MATRIX ALGEBRA FOR HOMOGENEOUS LINEAR ALGEBRAIC SYSTEM

WEEK 7: MATRIX ALGEBRA FOR HOMOGENEOUS LINEAR ALGEBRAIC SYSTEM

7.1 Solving Homogeneous System of Linear Equations

Multicomponent systems result in *n* set(s) of mathematical equations that must be solved simultaneously. It can be represented by the following matrix format: $[C]\{X\} = \{B\}$. If $\{B\} = \{0\}$, it is known as homogeneous system of linear equations.

(i) In this study, methods used to solve the total solution of $\{X\}$ by using the $[C]$ & zero $\{B\}$ is out of scope.

Linear Algebraic Equations	Coefficient Matrix, $[C]$	*Unknown,* $\{X\}$	Zero $\{B\}$
$0.5x_1 + 2.5x_2 - 9x_3 = 0$ $-4.5x_1 + 3.5x_2 - 2x_3 = 0$ $-8x_1 - 9x_2 + 22x_3 = 0$ *n=3 where 3 sets of eqns. are given to solve* x_1 *,* x_2 *and* x_3 *respectively.*	$\begin{bmatrix} 0.5 & 2.5 & -9 \\ -4.5 & 3.5 & -2 \\ -8 & -9 & 22 \end{bmatrix}$	$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$	$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

Depending on the coefficient matrix that represents any physical system or application.

- If $|C| = 0$ & $\{B\} = \{0\}$, then the solutions of $\{X\}$ due to initial/boundary conditions are non-zero/ non-trivial.
- If $|C| \neq 0$ & $\{B\} = \{0\}$, then the solutions of $\{X\}$ due to initial/boundary conditions are zero/ trivial.

(ii) In this study, the main focus is to find the characteristic of the system in terms of the eigenvalue, λ_i & the corresponding eigenvector, $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_i}$ where $i = 1,2,\ldots,n\ mode$. This is known as eigenvalue/ eigenvector problem.

$[A]\{x\}_i = [\lambda_i]\{x\}_i$

$([A]-\lambda_i[I])\{x\}_i = \{0\}$

where λ_i is one of the eigenvalue of the matrix [A] ;

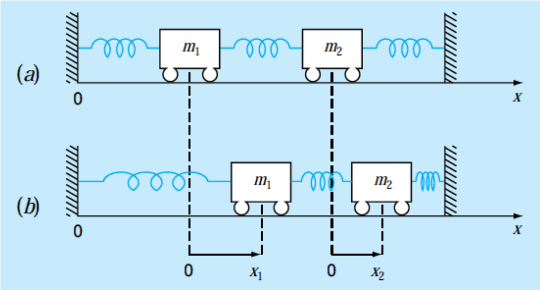
$\{x\}_i$ is the corresponding eigenvector for each λ_i and $\{x\}_i \neq \{0\}$, i.e. non-trivial solutions;

$[I]$= identity matrix

Note: In general, n dof system has n number of eigenvalue & eigenvector. For example: 2 dof mass-spring system has 2 eigenvalues and 2 eigenvectors, while 3 dof electrical circuit system has 3 eigenvalues and 3 eigenvectors.

7.2 Eigenvalue/Eigenvector Problem

Example: Link the eigenvalue and eigenvector to the characteristic of the given system.



Given $stiffness, k = k_1 = k_2 = 200N/m$;

$mass, m = m_1 = m_2 = 40kg$

The equations of motion of the 2 mass spring systems are provided:

$$-kx_1 - k(x_1 - x_2) = m_1\ddot{x}_1$$
$$k(x_1 - x_2) - kx_2 = m_2\ddot{x}_2$$
$$where\ x_1 = A_1 sin(\omega t + \theta_1)\ ,\ \ \ddot{x}_1 = -\omega^2 x_1$$
$$x_2 = A_2 sin(\omega t + \theta_2),\ \ \ddot{x}_2 = -\omega^2 x_2$$

$$\begin{bmatrix} \frac{2k}{m_1} - \omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{2k}{m_2} - \omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Note: The derivation of the eqns involves theory of vibration, thus it is not examined in this study.

$$\begin{bmatrix} 10 - \omega^2 & -5 \\ -5 & 10 - \omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 10 & -5 \\ -5 & 10 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 10 & -5 \\ -5 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} - \omega^2 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 10 & -5 \\ -5 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \omega^2 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$([A] - \lambda_i [I])\{x\}_i = \{0\}$$

$$\begin{bmatrix} A_{11} - \lambda_{mode\ i} & A_{12} \\ A_{21} & A_{22} - \lambda_{mode\ i} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_{mode\ i} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \lambda_{mode\ i} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_{mode\ i} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_{mode\ i} - \lambda_{mode\ i} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_{mode\ i} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_{mode\ i} = \lambda_{mode\ i} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_{mode\ i}$$

By comparing the general formulation of the eigenvalue/eigenvector problem,

$$[A]\{x\}_i = [\lambda_i]\{x\}_i$$

We find that the coefficient matrix, $[A] = \begin{bmatrix} 10 & -5 \\ -5 & 10 \end{bmatrix}$

Eigenvalue, $[\lambda_i] = \omega^2$ where ω =natural frequency of the system

Eigenvector, $\{x\}_i = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ = mode shape of the system (i.e. pattern of the maximum vibration amplitude) at the corresponding i^th mode/ case

Note: Natural frequency and mode shape are important characteristics for a vibration system that can be obtained from the eigenvalue/eigenvector problem.

Example: Solving the eigenvalue/eigenvector problem.

$$\begin{bmatrix} 10-\omega^2 & -5 \\ -5 & 10-\omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

To have non-trivial solution, $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \neq \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. The determinant must be zero.

$$\begin{vmatrix} 10-\omega^2 & -5 \\ -5 & 10-\omega^2 \end{vmatrix} = 0$$

Let the eigenvalue, $\lambda=\omega^2$

$$\begin{vmatrix} 10-\lambda & -5 \\ -5 & 10-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 20\lambda + 75 = 0$ Note: This is known as **characteristic equation**.

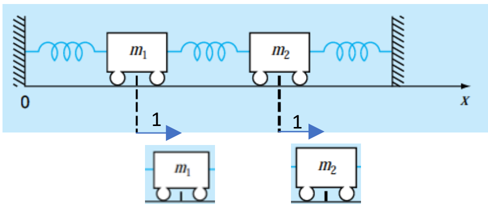
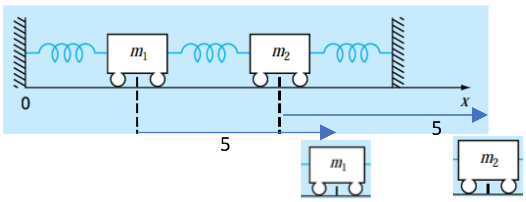
We obtain 2 eigenvalues for the 2 mass spring system: λ_1=5 and λ_2=15

Hint: Common practice is to arrange λ_i in ascending order, i.e. $\lambda_1 < \lambda_2$

Since $\lambda=\omega^2$, we can obtain the natural frequencies for the system: $\omega_1 = \sqrt{5}$ and $\omega_2 = \sqrt{15}$

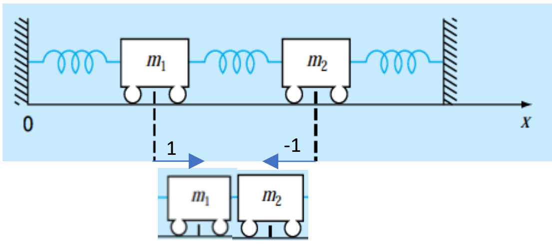
At mode 1, $\omega_1 = \sqrt{5}$ or λ_1=5, we obtain the unscaled eigenvector as following:

$$\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \xrightarrow{expand} \begin{matrix} 5x_1 - 5x_2 = 0 \\ -5x_1 + 5x_2 = 0 \end{matrix} \xrightarrow{at\ x_1 = 1, x_2 = 1\ for\ both\ eqns} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

	**Unscaled eigenvector** for mode #1 at $\omega_1 = \sqrt{5}$ $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ means that the maximum vibration of x_1 will be in phase with x_2, where both masses move to +x direction by one unit to the right at same time.
	**Unscaled eigenvector** for mode #1 at $\omega_1 = \sqrt{5}$ $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 5 \\ 5 \end{Bmatrix}$ is also an acceptable unscaled eigenvector answer. The most important point is eigenvector tells the **unique shape/ vibration pattern** at each mode regardless of the scale. In general, all acceptable solutions of eigenvector are called **eigenspace**, i.e. $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = t \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, where $-\infty \leq t \leq \infty$.
Normalized eigenvector has unique shape and unique scale by using the following formula: $\begin{Bmatrix} x_1, scale \\ x_2, scale \end{Bmatrix}_1 = \frac{1}{magnitude} \begin{Bmatrix} x_1, unscale \\ x_2, unscale \end{Bmatrix}_1 = \pm \frac{1}{\sqrt{1^2 + 1^2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \pm \frac{1}{\sqrt{5^2 + 5^2}} \begin{Bmatrix} 5 \\ 5 \end{Bmatrix} = \pm \begin{Bmatrix} 0.707 \\ 0.707 \end{Bmatrix}$	

At mode 2, $\omega_2 = \sqrt{15}\ or\ \lambda_2 = 15$, we obtain the unscaled eigenvector as following:

$$\begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \xrightarrow{expand} \begin{matrix} -5x_1 - 5x_2 = 0 \\ -5x_1 - 5x_2 = 0 \end{matrix} \xrightarrow{at\ x_1 = 1, x_2 = -1\ for\ both\ eqns} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$



Unscaled eigenvector for mode #2 at $\omega_2 = \sqrt{15}$

$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ means that at $\omega_2 = \sqrt{15} rads^{-1}$, the maximum vibration of x_1 will be out of phase with x_2, where one mass moves to +x direction while other to -x direction at same time.

Eigenspace for mode #2

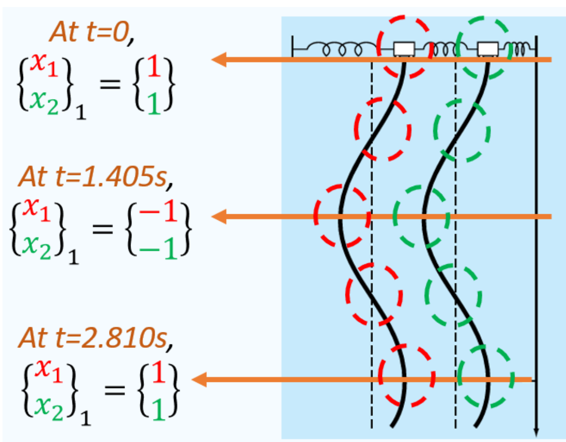
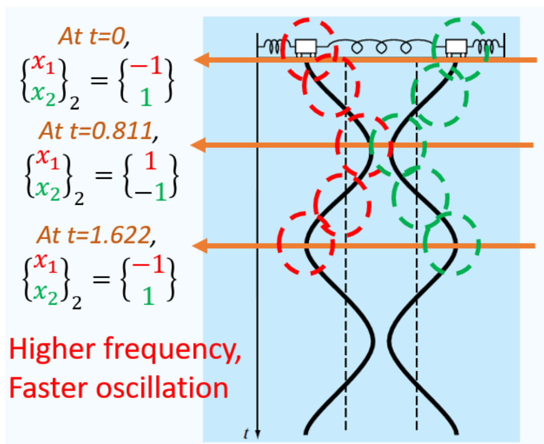
$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = t \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$, where $-\infty \le t \le \infty$.

Normalized eigenvector for mode #2

$$\begin{Bmatrix} x_1, scale \\ x_2, scale \end{Bmatrix}_2 = \frac{1}{magnitude} \begin{Bmatrix} x_1, unscale \\ x_2, unscale \end{Bmatrix}_2 = \pm \frac{1}{\sqrt{1^2 + 1^2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \pm \begin{Bmatrix} 0.707 \\ -0.707 \end{Bmatrix}$$

Note: Depending on questions, student should know how to find unscaled eigenvector/ eigenspace/ normalized eigenvector. In general, if it is not stated for *manual calculation*, then providing unscaled eigenvector (t =1) is sufficient. For *software calculation*, normalised eigenvector is used usually.

Visualisation of the eigenvalue & eigenvector information:

Mode 1/ Case 1	Mode 2/ Case 2
Eigenvalue, λ_1=5 Natural frequency, $\omega_1 = \sqrt{5}$ (Lower frequency) Unscaled eigenvector (also called mode shape), $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \pm \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$	λ_2=15 $\omega_2 = \sqrt{15}$ (Higher frequency) $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \pm \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$
	

Hint: Period = 1/ Frequency = $2\pi / \omega$

Example: Find the eigenvalues & eigenvectors of the following matrix.

$$\mathbf{A}=\begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix}$$

Eigenvalues/eigenvectors problem: $(\mathbf{A}-\lambda\mathbf{I})\boldsymbol{x}=\mathbf{0}$

$$\left(\begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

$$\begin{bmatrix}1-\lambda & -3 & 3\\ 3 & -5-\lambda & 3\\ 6 & -6 & 4-\lambda\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

Since $\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}\neq\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$, $\begin{vmatrix}1-\lambda & -3 & 3\\ 3 & -5-\lambda & 3\\ 6 & -6 & 4-\lambda\end{vmatrix}=0$

$$(1-\lambda)[(-5-\lambda)(4-\lambda)-3(-6)]-(-3)[3(4-\lambda)-3(6)]+3[3(-6)-(-5-\lambda)(6)]=0$$

Characteristic eqn: $\lambda^3-12\lambda-16=0$

$$(\lambda-4)\left(\lambda^2+4\lambda+4\right)=0$$

$\lambda_1=-2$, $\lambda_2=-2$ (Repeated eigenvalue case), $\lambda_3=4$

Case 1: $\lambda_1=-2$, Case 2: $\lambda_2=-2$ (Repeated eigenvalue case)

$$\begin{bmatrix}3 & -3 & 3\\ 3 & -3 & 3\\ 6 & -6 & 6\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}_{\lambda=-2}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

$$\xrightarrow{augmented}\left[\begin{array}{ccc:c}1 & -1 & 1 & 0\\ 3 & -3 & 3 & 0\\ 6 & -6 & 6 & 0\end{array}\right]$$

$$\xrightarrow[R_1\to\frac{R_1}{3}]{scale\ pivot\ element\ to\ 1}\left[\begin{array}{ccc:c}1 & -1 & 1 & 0\\ 3 & -3 & 3 & 0\\ 6 & -6 & 6 & 0\end{array}\right]$$

$$\xrightarrow[\substack{R_2\to R_2-3R_1\\ R_3\to R_3-6R_1}]{Forward\ elimination}\left[\begin{array}{ccc:c}1 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

Note: RREF shows rank 1 (i.e. 1 linearly independent vector)

$$x_1-x_2+x_3=0$$

$$x_1=x_2-x_3$$

$$Eigenspace\ for\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}_{\lambda=-2}=\begin{Bmatrix}x_2-x_3\\ x_2\\ x_3\end{Bmatrix}=\begin{Bmatrix}x_2\\ x_2\\ 0\end{Bmatrix}+\begin{Bmatrix}-x_3\\ 0\\ x_3\end{Bmatrix}=t\begin{Bmatrix}1\\ 1\\ 0\end{Bmatrix}\Bigg|_{x_2=t}+s\begin{Bmatrix}-1\\ 0\\ 1\end{Bmatrix}\Bigg|_{x_3=s},where\ t\&S\in\mathbf{R}$$

Eigenvectors, $\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}_{\lambda=-2}=\begin{Bmatrix}1\\ 1\\ 0\end{Bmatrix}\&\begin{Bmatrix}-1\\ 0\\ 1\end{Bmatrix}$ *for repeated eigenvalues* $\lambda_1=-2$, $\lambda_2=-2$ respectively.

Case 3: $\lambda_3 = 4$ (Distinct eigenvalue case)

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=4} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xrightarrow{augmented} \left[\begin{array}{ccc:c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\text{scale pivot element to 1}} \quad R_1 \to -\frac{R_1}{3} \quad \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\text{Forward elimination}} \quad R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 6R_1 \quad \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{\text{scale pivot element to 1}} \quad R_2 \to -\frac{R_2}{12} \quad \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -12 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{\text{Forward elimination}} \quad R_3 \to R_3 + 12R_2 \quad \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\text{Forward elimination}} \quad R_1 \to R_1 - R_2 \quad \left[\begin{array}{ccc:c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note: RREF shows rank 2 (i.e. 2 linearly independent vectors)

$$x_1 - \frac{1}{2}x_3 = 0 \;>>\; x_1 = \frac{1}{2}x_3$$

$$x_2 - \frac{1}{2}x_3 = 0 \;>>\; x_2 = \frac{1}{2}x_3$$

$$Eigenspace\ for \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=4} = \begin{Bmatrix} \frac{1}{2}x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{Bmatrix} = t \begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{Bmatrix}\Bigg|_{x_3=t} \quad ,where\ t \in \mathbf{R}$$

$$Eigenvector, \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=4} = \begin{Bmatrix} 0.5 \\ 0.5 \\ 1 \end{Bmatrix}$$

For verification of the eigenvector results, it should satisfy the eigenvalue/eigenvector problem:

$$[A]\{x\}_i = [\lambda_i]\{x\}_i$$

Case 1 ($\lambda = -2$)	Case 2 ($\lambda = -2$)	Case 3 ($\lambda = 4$)
$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = -2 \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$	$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = -2 \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$	$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0.5 \\ 0.5 \\ 1 \end{Bmatrix} = 4 \begin{Bmatrix} 0.5 \\ 0.5 \\ 1 \end{Bmatrix}$

Or we can combine all cases into single matrix operation:

Eigenvector/ Modal matrix consists of all eigenvectors, **P**	*Eigenvalue/ Spectral matrix consists of all eigenvalues,* **D**	Verification of Eigenvalue/ Eigenvector Problem for All Cases
$\mathbf{P}$ or $[P]$ $= \begin{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_1 & \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 & \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 \end{bmatrix}$ $= \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix}$ where $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_1 = eigenvector\ \#1$	$\mathbf{D}$ or $[D]$ $= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ $= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ where $\lambda_1 = eigenvalue\ \#1$	$[A]\{x\}_i = [\lambda_i]\{x\}_i\ \overrightarrow{extend}$ $[A][P] = [P][D]$ $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $= -2\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore Since\ LHS = RHS, [P] and\ [D] are\ verified.$

6.4 Engineering Application of Eigenvalue/Eigenvector Problem

(a) Diagonalization

- Eigenvectors are useful to diagonalize a square matrix:
- $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ if $|\mathbf{P}| \neq 0$ $\ where\ \mathbf{P} = full\ rank\ eigenvector\ or\ modal\ matrix$

Previously, eigenvalues/eigenvectors problem: $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0}$

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$Eigenvectors, \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=-2} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \& \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$$

$$Eigenvector, \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda=4} = \begin{Bmatrix} 0.5 \\ 0.5 \\ 1 \end{Bmatrix}$$

$\therefore$*Eigenvector matrix consists of all eigenvectors,* **P** $= \begin{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_1} & \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_2} & \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{\lambda_3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix}$

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note: We can convert a non-diagonal matrix $\mathbf{A}$ to a diagonal matrix, where the diagonal matrix, $\mathbf{D}$ consists of the eigenvalues of matrix $\mathbf{A}$ at the diagonal elements. $\lambda_1 = -2$, $\lambda_2 = -2$, $\lambda_3 = 4$

(b) Extension from Diagonalization

$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$

,where $\mathbf{A}$ can be expressed in terms of the eigenvector matrix, $\mathbf{P}$ and eigenvalue matrix, $\mathbf{D}$

$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) = (\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1})$ where $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$

$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = (\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) = (\mathbf{P}\mathbf{D}^3\mathbf{P}^{-1})$

$\vdots$

$\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ where $\mathbf{D}^k = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$, where $k \in \mathbf{R}$

(Comment: power of a diagonal matrix can be computed easily!)

Note: This formula implies that change of power of $\mathbf{A}$ will change the eigenvalue matrix while remain the eigenvector matrix, If $\mathbf{A}$ has eigenvalues of $\lambda_1, \lambda_2, \lambda_3$. Then, $\mathbf{A}^k$ has eigenvalues of $\lambda_1^k, \lambda_2^k, \lambda_3^k$. e.g. $\mathbf{A}^{-1}$ has eigenvalues of $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$.

You can find $\mathbf{A}^k$ (power of a matrix) with the eigenvalue & eigenvector matrix by using this formula.

$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$; Eigenvalue matrix, $\mathbf{D} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; Eigenvector matrix, $\mathbf{P} = \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix}$

$$\mathbf{A}^{100} = \mathbf{P}\mathbf{D}^{100}\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}^{100} \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^{100} & 0 & 0 \\ 0 & (-2)^{100} & 0 \\ 0 & 0 & (4)^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$= 10^{60} \times \begin{bmatrix} 0.8035 & -0.8035 & 0.8035 \\ 0.8035 & -0.8035 & 0.8035 \\ 1.6069 & -1.6069 & 1.6069 \end{bmatrix}$$

Some useful properties of eigenvalues, λ_i:

Trace($\mathbf{A}$) $= \sum \lambda_i$ $\rightarrow 1 - 5 + 4 = -2 - 2 + 4 = 0$

Determinant($\mathbf{A}$) $= \prod \lambda_i \rightarrow 1(-20 + 18) - (-3)(12 - 18) + 3(-18 + 30) = (-2)(-2)(4) = 16$

Eigenvalue ($\mathbf{A}$) = eigenvalue ($\mathbf{A}^T$) $= \lambda_i$ $\rightarrow \lambda_1 = -2; \lambda_2 = -2; \lambda_3 = 4$

Eigenvalue ($k\mathbf{A}$) $= k\lambda_i$,where $k \in \mathbf{R}$ $\rightarrow$ Eigenvalue$(5\mathbf{A}) = 5\lambda_1 = -10; 5\lambda_2 = -10; 5\lambda_3 = 20$

Eigenvalue ($\mathbf{A}^k$) $= \lambda_i^k$,where $k \in \mathbf{R}$ $\rightarrow$ Eigenvalue$(\mathbf{A}^5) = \lambda_1^5 = (-2)^5; \lambda_2^5 = (-2)^5; \lambda_3^5 = (4)^5$

Eigenvalue ($\mathbf{A} \pm k\mathbf{I}$) $= \lambda_i \pm k$ $\rightarrow$ Eigenvalue$(\mathbf{A} + 5\mathbf{I}) = \lambda_1 + 5 = 3; \lambda_2 + 5 = 3; \lambda_3 + 5 = 9$

(b) Cayley-Hamilton Theorem

- Characteristic equation of eigenvalue/eigenvector problem is useful to compute the power of matrix.

Previously, eigenvalues/eigenvectors problem: $(\mathbf{A}-\lambda\mathbf{I})\boldsymbol{x}=\mathbf{0}$

$$\mathbf{A}=\begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix} \rightarrow \qquad \begin{bmatrix}1-\lambda & -3 & 3\\ 3 & -5-\lambda & 3\\ 6 & -6 & 4-\lambda\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

- Characteristic eqn: $f(\lambda)=|(\mathbf{A}-\lambda\mathbf{I})|=0$, where λ=eigenvalue

$$p_0+p_1\lambda+p_2\lambda^2+\cdots+p_n\lambda^n=0$$

$$\lambda^3-12\lambda-16=0$$

- Cayley-Hamilton Theorem: $f(\mathbf{A})=p_0\mathbf{I}+p_1\mathbf{A}+p_2\mathbf{A}^2+\cdots+p_n\mathbf{A}^n=0$

,where $\mathbf{A}$ is the matrix that has the eigenvalue, λ. It shows that not only eigenvalue can satisfy the characteristic equation, but also the original coefficient matrix, **A.**

$$\mathbf{A}^3-12\mathbf{A}-16\mathbf{I}=\mathbf{0}\quad [Cayley-Hamilton\ Form]$$

<u>You can find $\mathbf{A}^n$ (power of a matrix) with the characteristic equation only by using this theorem. For example:</u>

$$\mathbf{A}^2+12\mathbf{I}-16\mathbf{A}^{-1}=\mathbf{0}\rightarrow \qquad \mathbf{A}^{-1}=\frac{1}{16}(\mathbf{A}^2+12\mathbf{I})$$

$$\mathbf{A}^3=12\mathbf{A}+16\mathbf{I}$$

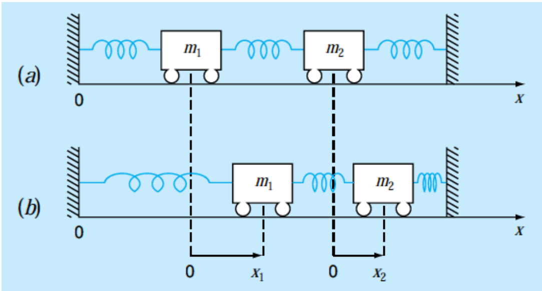
$$\mathbf{A}^4=12\mathbf{A}^2+16\mathbf{A}=12(-12\mathbf{I}+16\mathbf{A}^{-1})+16\mathbf{A}$$

$$\mathbf{A}^5=12\mathbf{A}^3+16\mathbf{A}^2=12(12\mathbf{A}+16\mathbf{I})+16(-12\mathbf{I}+16\mathbf{A}^{-1})$$

$$\vdots$$

6.5 Engineering Application of Solving Homogeneous System of Linear Equations

So far, we have determined the eigenvalue & eigenvector for a homogeneous system of linear equations. This information is useful to find the **eigenfunction** (i.e. Each of a set of independent functions which are the solutions to a given differential equation.)



Given $stiffness, k = k_1 = k_2 = 200N/m$;

$$mass, m = m_1 = m_2 = 40kg$$

The equations of motion of the 2 mass spring systems are provided:

$$-kx_1 - k(x_1 - x_2) = m_1\ddot{x}_1$$
$$k(x_1 - x_2) - kx_2 = m_2\ddot{x}_2$$
$$where\ x_1 = A_1 sin(\omega t + \theta_1)\ ,\ \ \ddot{x}_1 = -\omega^2 x_1$$
$$x_2 = A_2 sin(\omega t + \theta_2),\ \ \ddot{x}_2 = -\omega^2 x_2$$

$$\begin{bmatrix} 10-\omega^2 & -5 \\ -5 & 10-\omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Previously, solving the eigenvalue/eigenvector problem gives:

Eigenvalue: λ_1=5; λ_2=15

(where ω_1=$\sqrt{\lambda_1}$; ω_2=$\sqrt{\lambda_2}$)

Eigenvector: $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$; $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

Note: Finding eigenfunction or the solution of homogeneous system of linear equations is out of scope and it is including here for your extra info.

Eigenfunction #1 = $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 sin\left(\sqrt{\lambda_1}t + \theta_1\right) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} sin\left(\sqrt{5}t + \theta_1\right)$

Eigenfunction #2 = $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 sin\left(\sqrt{\lambda_2}t + \theta_2\right) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} sin\left(\sqrt{15}t + \theta_2\right)$

The total solution of the homogeneous linear algebraic system is equal to the superposition of all the eigenfunctions:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = c_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} sin\left(\sqrt{5}t + \theta_1\right) + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} sin\left(\sqrt{15}t + \theta_2\right)$$

,where c_1 & c_2 are unknown constants that can be obtained from the initial or boundary conditions. You will learn it in the ODE chapter later.

Verification of eigenfunction as the solution to the equations:

$$-200x_1 - 200(x_1 - x_2) = 40\ddot{x}_1$$

$$200(x_1 - x_2) - 200x_2 = 40\ddot{x}_2$$

Verification of eigenfunction #1: $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} sin(\sqrt{5}t + \theta_1)$; $\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{Bmatrix} -5 \\ -5 \end{Bmatrix} sin(\sqrt{5}t + \theta_1)$

LHS	RHS
$-200x_1 - 200(x_1 - x_2)$ $= -200(sin(\sqrt{5}t + \theta_1)\ ) - 200\left((sin(\sqrt{5}t + \theta_1)\ ) - (sin(\sqrt{5}t + \theta_1)\ )\right)$ $= -200(sin(\sqrt{5}t + \theta_1)\ )$	$40\ddot{x}_1$ $= 40\left(-5sin(\sqrt{5}t + \theta_1)\right)$ $= \left(-200sin(\sqrt{5}t + \theta_1)\right)$
$200(x_1 - x_2) - 200x_2$ $= 200\left((sin(\sqrt{5}t + \theta_1)\ ) - (sin(\sqrt{5}t + \theta_1)\ )\right) - 200(sin(\sqrt{5}t + \theta_1)\ )$ $= -200(sin(\sqrt{5}t + \theta_1)\ )$	$40\ddot{x}_2$ $= 40\left(-5sin(\sqrt{5}t + \theta_1)\right)$ $= \left(-200sin(\sqrt{5}t + \theta_1)\right)$

$\therefore$ *Since LHS = RHS*, thus it is proven that eigenfunction #1 is one of the solution

Verification of eigenfunction #2: $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} sin(\sqrt{15}t + \theta_2)$; $\begin{Bmatrix} -15 \\ 15 \end{Bmatrix} sin(\sqrt{15}t + \theta_2)$

LHS	RHS
$-200x_1 - 200(x_1 - x_2)$ $= -200\left(sin(\sqrt{15}t + \theta_2)\right) - 200\left((sin(\sqrt{15}t + \theta_2)\ ) - (-sin(\sqrt{15}t + \theta_2)\ )\right)$ $= -600(sin(\sqrt{15}t + \theta_2)\ )$	$40\ddot{x}_1$ $= 40\left(-15sin(\sqrt{15}t + \theta_2)\right)$ $= \left(-600sin(\sqrt{15}t + \theta_2)\right)$
$200(x_1 - x_2) - 200x_2$ $= 200\left((sin(\sqrt{15}t + \theta_2)\ ) - (-sin(\sqrt{15}t + \theta_2)\ )\right) - 200(-sin(\sqrt{15}t + \theta_1)\ )$ $= 600(sin(\sqrt{15}t + \theta_1)\ )$	$40\ddot{x}_2$ $= 40\left(15sin(\sqrt{15}t + \theta_1)\right)$ $= \left(600sin(\sqrt{15}t + \theta_1)\right)$

$\therefore$ *Since LHS = RHS*, thus it is proven that eigenfunction #2 is one of the solution