ENGINEERING APPLICATION OF INTEGRALS

WEEK 9: ENGINEERING APPLICATION OF INTEGRALS

9.1 (I) AREA BETWEEN CURVES

In this section we will determine the area of a region between two curves by integrating with respect to the independent variable. The area of a region between two curves will also be determined using definite integration with respect to the dependent variable. We also try to find the area of a compound region.

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a,b]$ such that $f(x)\geq g(x)$ on $[a,b]$. We want to find the area between the graphs of the functions, as shown in Figure 9.1.

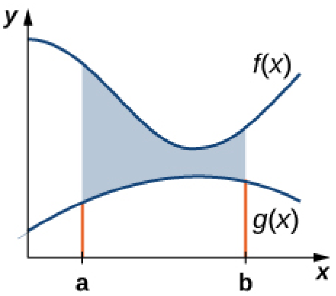


Figure 9.1: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a,b]$

As we did before, we are going to partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i=0,1,2,\ldots,n$, let $P=x_i$ be a regular partition of $[a,b]$. Then, for $i=1,2,\ldots,n$, choose a point $x^*_i\in[x_{i-1},x_i]$, and on each interval $[x_{i-1},x_i]$ construct a rectangle that extends vertically from $g(x^*_i)$ to $f(x^*_i)$. Figure 9.2(a) shows the rectangles when x^*_i is selected to be the left endpoint of the interval and $n=10$. Figure 9.2(b) shows a representative rectangle in detail.

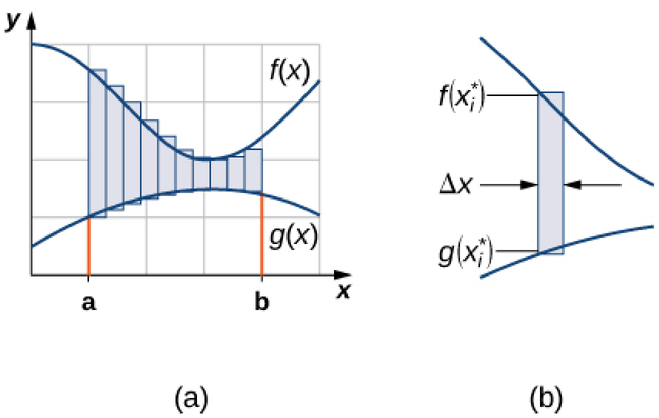


Figure 9.2 : (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x^*_i) - g(x^*_i)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by:

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x \qquad (9.1)$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get,

$$A = \lim_{n\to\propto} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]dx \qquad (9.2)$$

In conclusion, let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a,b]$. Let "R" denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x=a$ and $x=b$, respectively. Then, the area of "R" is given by,

$$A = \int_a^b [f(x) - g(x)]dx \qquad (9.3)$$

EXAMPLE 9.1

Find the area bounded by two curves, $f(x) = -x^2 + 4x + 3$ and $g(x) = -x^3 + 7x^2 - 10x + 5$ over the interval $1 \leq x \leq 2$.

Solution:

The region can be depicted as shown in Figure 9.3(a) with the desired area shaded. Also, we can depict f alone with the area under f shaded, and then g alone with the area under g shaded.

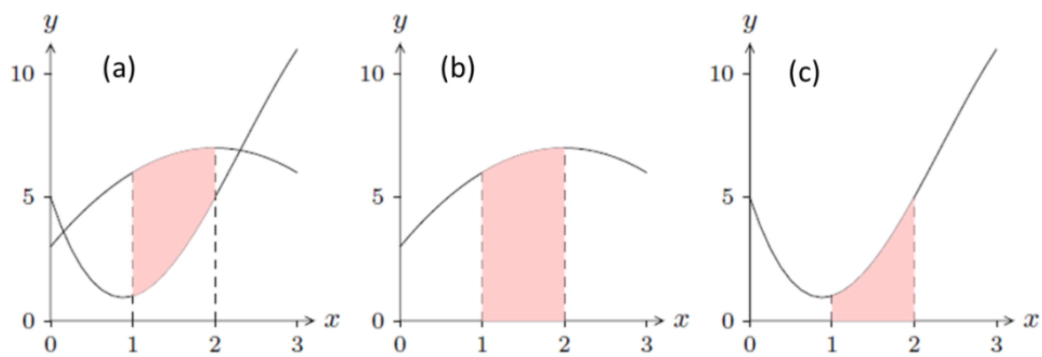


Figure 9.3: Area between curves as a difference of areas

It is clear from the figure that the area we want is the area under f minus the area under g, which can be written as,

$$A = \int_1^2 f(x)dx - \int_1^2 g(x)dx = \int_1^2 [f(x) - g(x)]dx$$

$$= \int_1^2 [(-x^2 - x - 3) - (-x^3 + 7x^2 - 10x + 5)]dx$$

$$= \frac{x^4}{4} - \frac{8x^3}{3} + 7x^2 - 2x\Big|_1^2$$

$$= \frac{49}{12}$$

EXAMPLE 9.2

If R is the region bounded above by the graph of the function f(x) = 9−(x/2)^2 and below by the graph of the function g(x)=6−x, find the area of region R.

Solution:

The region can be depicted by the following figure.

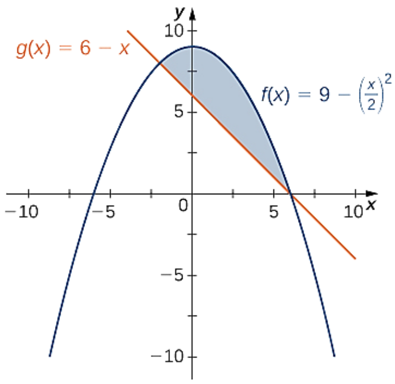


Figure 9.4 : This graph shows the region below the graph of f(x) and above the graph of g(x).

We first need to compute where the graphs of the functions intersect. Setting f(x)=g(x), we get,

$$f(x) = g(x)$$

$$9 - (\frac{x}{2})^2 = 6 - x$$

$$36 - x^2 = 24 - x$$

$$(x - 6)(x + 2) = 0$$

$$x = (6, -2)$$

The graphs of the functions intersect when x=6 or x=−2, so we want to integrate from −2 to 6 . Since f(x)≥g(x) for −2≤x≤6, we obtain,

$$A = \int_a^b [f(x) - g(x)]\, dx$$
$$= \int_{-2}^{6} \left[9 - (\frac{x}{2})^2 - (6 - x)\right] dx$$
$$= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx$$
$$= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Big|_{-2}^{6} = \frac{64}{3}.$$

9.1 (II) AREAS OF COMPOUND REGIONS

So far, we have required, $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

EXAMPLE 9.3

Finding the Area of a Region Bounded by Functions That Cross

If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0,\pi]$, find the area of region R.

The region can be depicted by the following figure.

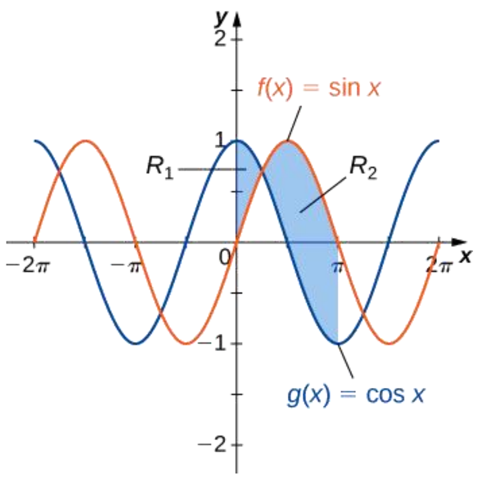


Figure 9.5 : The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x=\pi/4$. For $x\in[0,\pi/4]$, $\cos x \geq \sin x$, thus,

$$|f(x) - g(x)| = |sinx - cosx| = cosx - sinx$$

On the other hand, for $x\in[\pi/4,\pi]$, $\sin x \geq \cos x$, thus,

$$|f(x) - g(x)| = |sinx - cosx| = sinx - cosx$$

Now,

$$A = \int_a^b [f(x) - g(x)]dx$$

$$= \int_0^{\pi} [sinx - cosx]dx$$

$$= \int_0^{\pi/4} [cosx - sinx]dx + \int_{\pi/4}^{\pi} [sinx - cosx]dx$$

$$= (\sqrt{2} - 1) + (1 + \sqrt{2})$$

$$= 2\sqrt{2}$$

EXAMPLE 9.4

Consider the region depicted in Figure 9.6 . Find the area of R .

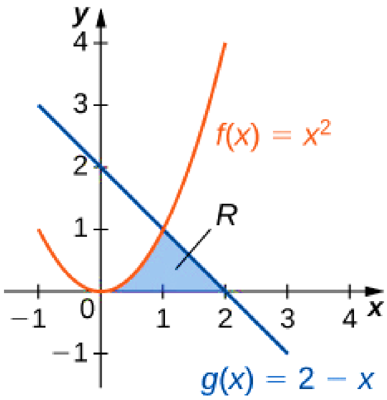


Figure 9.6 : Two integrals are required to calculate the area of this region.

As with Example 9.3, we need to divide the interval into two pieces. The graphs of the functions intersect at $x=1$ (set $f(x)=g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval [0,1] and one over the interval [1,2] .

Over the interval [0,1] , the region is bounded above by $f(x)=x^2$ and below by the x-axis, so we have,

$$A1 = \int_0^1 x^2 dx = \frac{x^3}{3} = \frac{1}{3}$$

Over the interval [1,2], the region is bounded above by $g(x)=2-x$ and below by the x-axis, so we have,

$$A2 = \int_1^2 (2 - x)dx = \frac{1}{2}$$

Adding these areas together, we obtain, $A = A1+A2 = 5/6$

The area of the region is, $5/6$ units^2.

9.2 HYDROSTATIC FORCE

In this section, we will learn about the force exerted by a fluid on a horizontal surface. Also, we will determine the hydrostatic force against a vertical surface.

9.2 (I) FORCE EXERTED BY A FLUID ON A HORIZONTAL SURFACE

Deep-sea divers know that pressure increases as they swim deeper because their bodies have to support larger volumes of water. If a planar plate with an area of A square meters is submerged horizontally in a fluid at a depth of d meters, as in Figure 9.7, then the volume of the fluid above the plate is $V = Ad$. Moreover, the mass m of the fluid above the plate is $m = V\omega$ where ω is the mass density of the fluid in kilograms per cubic meter. Applying Newton's Second Law of Motion, the force exerted by the fluid on the plate is:

$$\text{Force} = mg = Ad\omega g$$

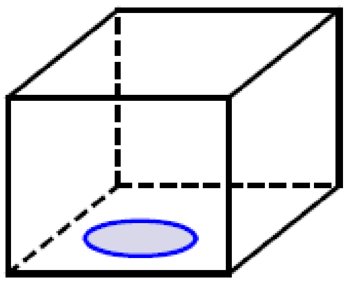


Figure 9.7: A plate of area A is submerged horizontally in a fluid of depth d.

Where, g is the acceleration due to gravity. The product $\omega = \omega g$ is called the weight density of the fluid, and is the weight per unit volume of the fluid. Then the force exerted by the fluid on the plate at a depth d meters is given by:

$$\text{Force} = (\text{Area}) \cdot (\text{Depth}) \cdot (\text{Weight Density}) = Ad\omega'$$

Consequently, the pressure or force per unit area weighing on the horizontal plate is defined as:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = d\omega'$$

The above equation implies that the pressure is directly proportional to the depth. This is why a deep-sea diver experiences more pressure as he or she swims deeper.

9.2 (II) HYDROSTATIC FORCE AGAINST A VERTICAL SURFACE

Early in the section, we described the force exerted by a fluid on a horizontal surface without using calculus, see Definition 8. However, in order to find the force exerted by a fluid on a vertical surface we have to apply calculus. In addition, we apply a physical principle due to Blaise Pascal, i.e., at a fixed depth a fluid exerts the same pressure in all directions.

We draw a y-axis so that the depths of a vertical surface vary from points y = c to y = d on the y-axis. Then partition [c, d] into n subintervals of equal width Δy. If Δy is small, the portion of the vertical plate associated to the i_{th} subinterval $[y_{i-1}, y_i]$ is approximately a vertical rectangular strip R_i of length $L(y_i)$ and width Δy. Moreover, the depth of R_i from the surface of the fluid is almost constant and which we denote by $d(y_i)$. Applying the above principle due to Pascal, the force F_i exerted by the fluid on a vertical rectangle R_i is approximately:

$$F_i \approx Area.Depth.Weight\ density$$

$$\approx [L(y_i)\Delta y]dy_i\omega'$$

We find it reasonable to believe that the sum of the Fi's approximates the force F exerted by the fluid on the entire vertical surface, i.e.,

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} [L(y_i)\Delta y]dy_i\omega'$$

Suppose a y-axis is positioned vertically so that the depths of the fluid vary from points y = c to y = d on the y-axis. At point y in [c, d], let d(y) and L(y) be the depth and length from side to side of the surface, respectively. The force exerted by the fluid on one side of the vertical surface is given by:

$$F = \int_c^d L(y).d(y).dy \qquad (9.4)$$

provided d(y) and L(y) are continuous functions of y.

EXAMPLE 9.5

A vertical dam has a cross section that is a trapezoid, see Figure 9.8. The dimensions of the trapezoid are 10 meters high, 100 meters across the top, and the base is 60 meters. If the dam is filled with water, find the force exerted by the water on the cross section.

Solution:

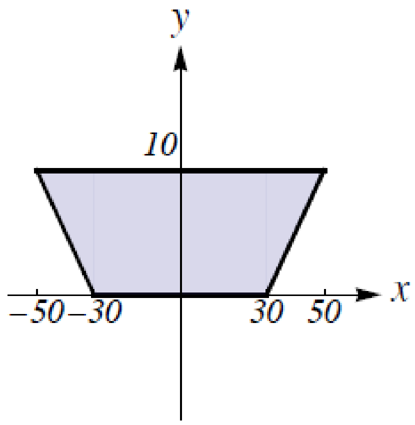


Figure 9.8: The cross section of a vertical dam.

Since water has a mass density of 1000 kg/m^3 and gravity is 9.8 m/sec^2, the weight density of water is:

$$\omega' = 1000.9.8 = 9800\ \frac{kg}{m^2 sec^2}$$

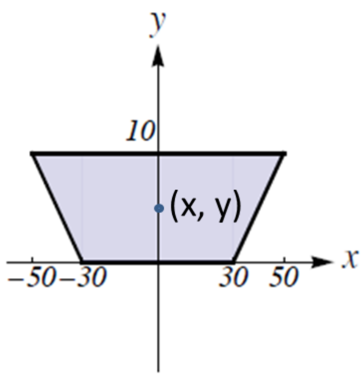


Figure 9.9: The cross section of a vertical dam

Let consider (x, y) is the center of the base of the dam. An equation of the slanted side of the dam in the first quadrant is:

$$y = \frac{x}{2} - 15$$

$$x = 2y + 30$$

At point y in [0, 10], the horizontal distance across the dam is:

$$L(y) = 2x = 4y + 60$$

and the water depth is: $d(y) = 10 - y$

Now, $Force = \omega' \int_c^d L(y)d(y)dy = 9800 \int_0^{10} (4y + 60)(10 - y)dy$

$= 3.593 \times 10^7 \frac{kg.m}{sec^2}$

EXAMPLE 9.6

A dam has the shape of the trapezoid shown in Figure 9.10. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

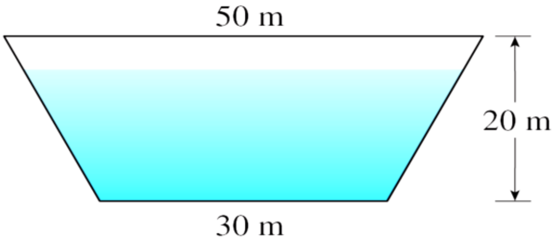


Figure 9.10: A dam with shape of trapezoid

Solution:

- Choose a vertical *x*-axis with origin at the surface of the water and directed downward
- Divide the depth of water into intervals between [0,16] of equal length
- At the *i*th strip $(x = x_i^*)$,

$$\frac{a}{16 - x_i^*} = \frac{10}{20}$$

From which,

$$a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2}$$

Width of the dam at the ith strip, w_i can be found by,

$$w_i = 2(15 + a) = 2\left(15 + 8 - \frac{x_i^*}{2}\right) = 46 - x_i^*$$

- At the ith strip, the area A_i can be approximated by,

$$A_i \approx w_i \Delta x = (46 - x_i^*)\Delta x$$

- When Δx is small, the pressure acting on the ith strip, P_i is almost constant:

$$P_i \approx 1000 g x_i^*$$

- The hydrostatic force on the ith strip, F_i,

$$F_i = P_i A_i \approx 1000 g x_i^* (46 - x_i^*) \Delta x$$

- Adding these forces and taking the limit as $n \to \infty$,

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 1000 g x_i^* (46 - x_i^*) \Delta x$$

$$= \int_0^{16} 1000 g x (46 - x) dx$$

$$= 1000(9.8) \int_0^{16} (46x - x^2) dx$$

$$= 9000 \left[23x^2 - \frac{x^3}{3} \right] \Big|_0^{16}$$

$$\approx 4.43 \ x \ 10^7 \ N$$

EXAMPLE 9.7

A circular plate with a radius of 1 ft is submerged vertically in a tank filled with water. The center of the plate is 4 ft from the surface of the water. Find the force exerted by the water on one side of the plate.

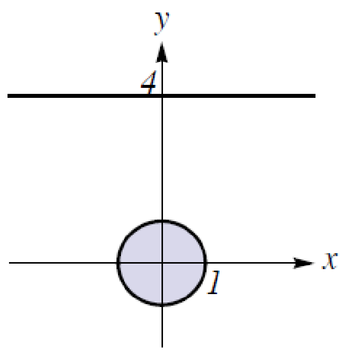


Figure 9.11: A plate with a circular cross section is submerged vertically.

Solution:

Draw a coordinate system where origin is at the center of the circle, see Figure 9.11. Then an equation of the circular plate of radius 1 is:

$$x^2 + y^2 = 1$$

Partition the interval [−1, 1] into smaller subintervals of equal width Δy. The portion of the circle in the i_{th} subinterval $[y_{i-1}, y_i]$ can be approximated by a rectangle R_i of length:

$$L(y_i) = 2\sqrt{(1 - y_i)^2}$$

The distance of any point in Ri from the surface of the water is approximately:

$$d(y_i) = 4 - y_i$$

Now, $Force = \ \omega' \int_c^d L(y)d(y)dy = 62.4 \int_{-1}^{1} (4 - y).\, 2\sqrt{(1 - y_i)^2}dy$

$$= 124.8 \int_{-1}^{1} [4\sqrt{(1 - y_i)^2} - y\sqrt{(1 - y_i)^2}]dy$$

$$= 124.8 \int_{-1}^{1} 4\sqrt{(1 - y_i)^2}dy - 128.4 \int_{-1}^{1} y\sqrt{(1 - y_i)^2}\, dy$$

In the above equation, the first integral is the area of a semicircle of radius 1 which is π/2, and second integral is zero since we are integrating an odd function over [−1, 1]. Then the force is:

$$Force = 124.8\, .4 . \frac{\pi}{2} = \ 249.6\pi \ \approx \ 784\ lb.$$

9.3 DISTANCE, VELOCITY, ACCELERATION

First recall a general principle that will later be applied to distance-velocity-acceleration problems, among other things. If F(u) is an anti-derivative of f(u), then:

$$\int_a^b f(u)du = F(b) - F(a)$$

Suppose that we want to let the upper limit of integration vary, i.e., we replace b by some variable x. We think of a as a fixed starting value x0. In this new notation the last equation (after adding F(a) to both sides) becomes:

$$F(x) = F(x0) + \int_{xo}^{x} f(u)du$$

(Here u is the variable of integration, called a "dummy variable," since it is not the variable in the function F(x). In general, it is not a good idea to use the same letter as a variable of integration and as a limit of integration. That is, $\int_{xo}^{x} f(x)dx$ is bad notation, and can lead to errors and confusion.)

An important application of this principle occurs when we are interested in the position of an object at time t (say, on the x-axis) and we know its position at time t0. Let s(t) denote the position of the object at time t (its distance from a reference point, such as the origin on the x-axis). Then the net change in position between t0 and t is s(t)−s(t0).

Since s(t) is an anti-derivative of the velocity function v(t), we can write:

$$s(t) = s(t0) + \int_{t0}^{t} v(u)du$$

Similarly, since the velocity is an anti-derivative of the acceleration function a(t), we have:

$$v(t) = v(t0) + \int_{t0}^{t} a(u)du$$

EXAMPLE 9.8

Suppose an object is acted upon by a constant force F. Find v(t) and s(t).

Solution:

By Newton's law F = ma, so the acceleration is F/m, where m is the mass of the object. Then we first have:

$$v(t) = v(t0) + \int_{t0}^{t} \frac{F}{m} du$$

$$v(t) = vo + \frac{F}{m} u|_{t0}^{t}$$

$$v(t) = v0 + \frac{F}{m}(t - t0)$$

Using the usual convention v0 = v(t0). Then:

$$s(t) = s(t0) + \int_{t0}^{t} [v0 + \frac{F}{m}(u - t0)]du$$

$$\boldsymbol{s(t) = s0 + v0(t - t0) + \frac{F}{2m}(t - t0)^2}$$

In the common case that t0 = 0, then:

$$\boldsymbol{s(t) = s0 + v0t + \frac{F}{2m} t^2}$$

EXAMPLE 9.9

The acceleration of an object is given by a(t) = cos(πt), and its velocity at time t = 0 is 1/(2π). Find both the net and the total distance travelled in the first 1.5 seconds.

Solution:

We compute:

$$v(t) = v0 + \int_0^t \cos(\pi u)\, du$$

$$= \frac{1}{2\pi} + \frac{1}{\pi}\sin(\pi u)|_0^t$$

$$= \frac{1}{2\pi} + \frac{1}{\pi} sin\pi t = \frac{1}{\pi}(\frac{1}{2} + sin\pi t)$$

The net distance travelled is then:

$$s(3/2) - s(t0) = \int_0^{3/2} (\frac{1}{2\pi} + \frac{1}{\pi} sin\pi t)dt$$

$$= [\frac{t}{2\pi} - \cos(\pi t)]|_0^{3/2}$$

$$= \frac{3}{4\pi} + \frac{1}{\pi^2}$$

$$\approx 0.340\ meters$$

To find the total distance travelled, we need to know when [0.5 + sin(πt)] is positive and when it is negative. This function is 0 when sin(πt) is −0.5, i.e., when πt = 7π/6, 11π/6, etc. The value πt = 7π/6, i.e., t = 7/6, is the only value in the range 0 ≤ t ≤ 1.5. Since v(t) > 0 for t < 7/6 and v(t) < 0 for t > 7/6, the total distance travelled is:

$$s(total) = \int_0^{7/6} \left(\frac{1}{2\pi} + \frac{1}{\pi} sin\pi t\right) dt + \left|\int_{7/6}^{3/2} (\frac{1}{2\pi} + \frac{1}{\pi} sin\pi t)dt\right|$$

$$\approx 0.409\ meters$$

9.4 WORK

In everyday life, work is a physical or mental activity that results in the completion of a certain task. The technical meaning of work in physics involves the concept of force acting on an object and the displacement of the object due to the force. Intuitively, force describes how an object is pushed or pulled.

For example, if a 150 kg person seats on top of a vertical spring, we say a force of 150 kg is acting on the top of a spring and the direction of the force is downward. Further, if the 150 kg force compresses the spring by 2 m, we say that the work done is the product of the force and the amount by which the spring is compressed. That is, if an object is moved in a straight line against a force F for a distance s the work done is W = Fs. In above case, W = 150 x 2 = 300 Kg-m.

Other examples of work include a student lifting her book sack, a gardener pushing a lawn mower, and a space shuttle lifting off for space. In reality few situations are very simple. However, the force might not be constant over the range of motion, and then we need to take help from integral.

EXAMPLE 9.10

How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit 100 miles above the surface?

Solution:

Over 100 miles the force due to gravity does change significantly, so we need to take this into account. The force exerted on a 10 pound weight at a distance r from the center of the earth is $F = k/r^2$ and by definition it is 10 when r is the radius of the earth (we assume the earth is a sphere). How can we approximate the work done? We divide the path from the surface to orbit into n small subpaths. On each subpath the force due to gravity is roughly constant, with value k/r^2_i at distance r_i. The work to raise the object from r_i to r_{i+1} is thus approximately $k/r^2{}_i\Delta r$ and the total work is approximately:

$$\sum_{i=0}^{n-1} \frac{k}{r_i^2}\Delta r$$

or in the limit, $W = \int_{ro}^{r1} \frac{k}{r^2} dr$

where r0 is the radius of the earth and r1 is r0 plus 100 miles. The work is,

$$W = \int_{ro}^{r1} \frac{k}{r^2} dr = -\frac{k}{r}\bigg|_{r0}^{r1} = -\frac{k}{r} + \frac{k}{r0}$$

Using r0 = 20925525 feet we have r1 = 21453525. The force on the 10 pound weight at the surface of the earth is 10 pounds, so $10 = k/20925525^2$, giving k = 4378775965256250.

Then,

$$-\frac{k}{r} + \frac{k}{r0} = \frac{491052320000}{95349} \approx 5150052\, ft - pounds$$

Note that if we assume the force due to gravity is 10 pounds over the whole distance we would calculate the work as 10.(r1−r0) = 10 · 100 · 5280 = 5280000, somewhat higher since we don't account for the weakening of the gravitational force.

EXAMPLE 9.11 (HOOKE'S LAW)

A force of 200 lb compresses a spring by a length of 0.5 ft from its natural length of 4 ft.

a) Find the spring constant.

b) Find the work needed to compress the spring from a length of 3.5 ft to a length of 3 ft.

Solution:

(a) Hooke's Law states the force satisfies

$$f(x) = kx$$

where k is a positive constant called the spring constant and x is not too big to cause the spring to break.

Now, we find that the spring constant k is,

$$k(0.5 \text{ ft}) = 200 \text{ lb}$$

$$k = 400 \text{ lb/ft}.$$

(b) By Hooke's Law, the force needed to compress the spring by a length of x feet from its natural length is

$$f(x) = kx = 400x$$

Since, k = 400. As the spring compresses from a length of 3.5 ft to a length of 3 ft, we can think of one end of the spring as moving along the x-axis from x = 0.5 to x = 1 while the other end of the spring is held fixed. Then the work done is,

$$W = \int_{0.5}^{1} 400x dx = \left.\frac{400x^2}{2}\right|_{0.5}^{1} = 150\ ft - lb$$

EXAMPLE 9.12 (WINDING A CABLE)

A 20 ft cable weighing 3 lb/ft is hanging from a winch. Find the work done by the winch in winding up all the cable.

Solution:

First we consider that the cable as an inverted vertical number line with the winch at the origin and the lower end of the cable at y = 20, see Figure 1. Subdivide the chain into smaller sections by partitioning [0, 20] into subintervals [yi−1, yi] of equal width Δy.

Since the cable weighs 3 lb/ft, the weight of the section of the chain in $[y_{i-1}, y_i]$ is 3Δy lb. If Δy is small, y_i is approximately y_{i-1}, and consequently the distance between the origin and the section $[y_{i-1}, y_i]$ is approximately y_i. Then the work W_i needed to lift the section of the chain in $[y_{i-1}, y_i]$ to the winch is approximately.

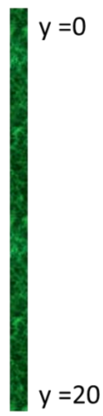


Figure 9.12 A 20 ft cable with the top at y = 0 and the bottom at y = 20

W_i = (force) (distance) ≈ (3Δy lb) (y_i ft) = $3y_i\Delta y$ ft-lb.

As the norm ||Δ|| of the partition of [0, 20] approaches zero, we find that the total work needed to lift the entire chain is given by,

$$W = \lim_{\Delta y \to 0} \sum_{i=1}^{n} 3y_i \ \Delta y = \int_0^{20} 3ydy = 600\ ft - lb$$

EXAMPLE 9.13 (WORK DONE IN PUMPING WATER)

An inverted circular cone with a height 6 ft and base radius 2 ft is filled with water to a depth of 4 ft, see Figure 9.13. Find the work needed to pump all the water to the top of the conical tank. The density of water is 62.4 lb/ft^3.

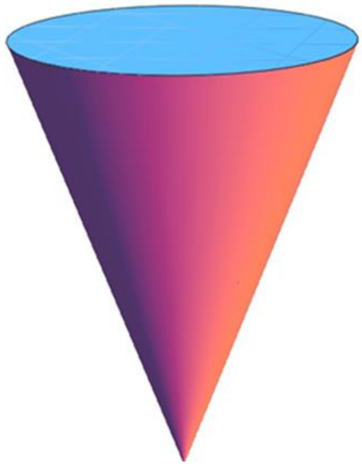


Figure 9.13: A conical water tank of height 6ft, base diameter 4ft, filled with water to a depth of 4ft.

Imagine inserting a vertical number line through the vertex of the cone with the origin at the top of the conical tank. Since the water is 4 ft deep, the water marks on the number line will lie on the interval [2, 6]. Subdivide [2, 6] into n subintervals [y_{i-1}, y_i] of equal width Δy. The volume of water in the depth [y_{i-1}, y_i] can be approximated by a circular disk Di of height Δy and radius r_i. In Figure 9.14, we see similar triangles and consequently.

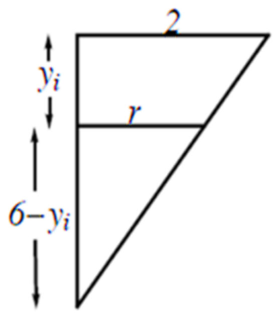


Figure 9.14 Similar triangle

$$\frac{r_i}{6 - y_i} = \frac{2}{6}$$

$$r_i = \frac{2}{6}\,(6 - y_i)$$

Then the volume V_i of disk D_i is,

$$V_i = \pi r^2 \Delta y = \frac{\mu}{9}(6 - y_i)^2 \Delta y$$

and using the density of water we find that the weight F_i of disk D_i is,

$$F_i = 62.4V_i = \frac{62.4\pi}{9}(6 - y_i)^2\Delta y$$

Then the work Wi needed to pump disk Di to the top of the tank is approximately,

$$W_i \approx (force)(distance) = F_i y_i = \frac{62.4\pi}{9} y_i(6 - y_i)^2\Delta y$$

Applying the limit process, we find that the total work needed to pump the water to the top of the conical tank is,

$$\begin{aligned}
(\text{Work}) &= \lim_{\Delta y \to 0} \sum_{i=1}^{n} \frac{62.4\pi}{9} y_i (6 - y_i)^2 \Delta y \\
&= \int_2^6 \frac{62.4\pi}{9} y (6 - y)^2 \, dy \\
&= \int_2^6 \frac{62.4\pi}{9} (y^3 - 12y^2 + 36y) \, dy \\
(\text{Work}) &= \frac{6656\pi}{15} \text{ ft-lb} \approx 1394 \text{ ft-lb.}
\end{aligned}$$

9.5 MOMENTS AND CENTRES OF MASS

Suppose a beam is l meters long, and that there are two weights on the beam: a m1 kilogram weight d1 meters from the fulcrum, another m2 kilogram weight d2 meters from the fulcrum end as in figure 9.8. Where a fulcrum should be placed so that the beam balances? Let's assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as x coordinates; the weights are at x = 3, x = 6, and x = 8, as in figure 9.15.

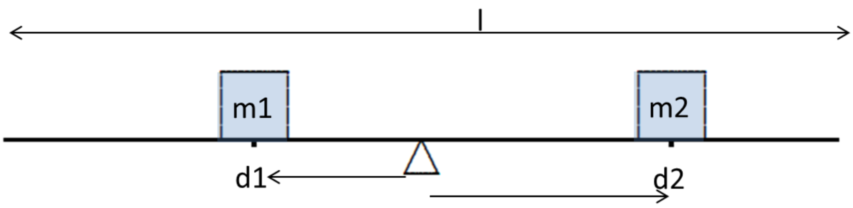


Figure 9.15: A beam with two masses.

Two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum. The rod will balance if $m_1 d_1 = m_2 d_2$.

Considering the following Figure 9.9, we can find out the position fulcrum.

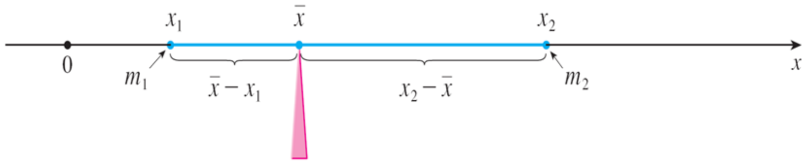


Figure 9.16

- suppose that the rod lies along the *x*-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\bar{x}$
- Compare with the previous figure,

$$d_1 = \bar{x} - x_1 \text{ and } d_2 = x_2 - \bar{x}$$

- Thus,

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

- In general, for a system with n particles with the mass of each particle $m_1,\ m_2\ \ldots,\ m_n$ located at points $x_1,\ x_2\ \ldots,\ x_n$ on the x-axis,
-

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

- If we let $m = \sum_{i=1}^{n} m_i$ and the sum of individual moments $M = \sum_{i=1}^{n} m_i x_i$, we obtain

$$m\bar{x} = M$$

We can consider another system, like Figure 9.17 below.

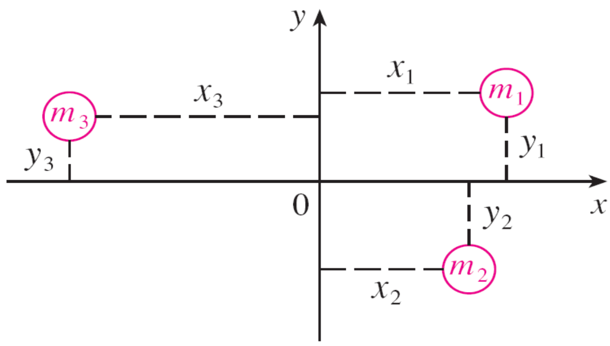


Figure 9.17

- consider a system of n particles with masses $m_1,\ m_2\ \dots,\ m_n$ located at the points $(x_1, y_1), (x_2, y_2),\ \dots, (x_n, y_n)$ in the xy-plane
- By analogy with the one-dimensional case, we define the **moment of the system about the *y*-axis** to be,

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system about the *x*-axis** as,

$$M_x = \sum_{i=1}^{n} m_i y_i$$

- the coordinates $(\bar{x}, \bar{y})$ of the center of mass are given in terms of the moments by the formulas,

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

- the center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass *m* would have the same moments as the system

EXAMPLE 9.14

Find the moments and centre of mass of the system of objects that have masses 3, 4, and 8 at the points (–1, 1), (2, –1), and (3, 2), respectively.

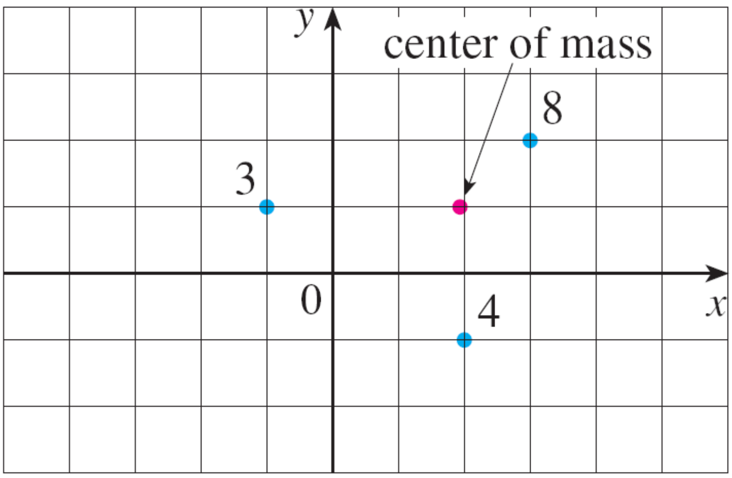


Figure 9.18

- Calculate the moments about x- and y-axes from the formula M_x and M_y
- Calculate the coordinate for the centre of mass $(\bar{x}, \bar{y})$ from the formula $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$

Solution:

The system of 3 particles with masses 3, $4\ and\ 8$ located at the points (–1, 1), (2, –1), and (3, 2), respectively, in the xy-plane.

The **moment of the system about the *x*-axis** as,

$$M_x = \sum_{i=1}^{n} m_i y_i = (3\text{x}1+4\text{x-}1+8\text{x}2) = 15$$

The **moment of the system about the *y*-axis** to be,

$$M_y = \sum_{i=1}^{n} m_i x_i = (3\text{x-}1+ 4\text{x}2+8\text{x}3) = 29$$

The center of mass $(\bar{x}, \bar{y})$ from the formula $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$

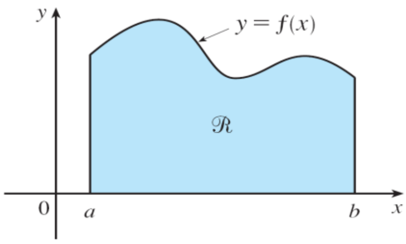
$$\bar{x} = \frac{M_y}{m} = \frac{29}{(3+4+8)} = 1.93$$

$$\bar{y} = \frac{M_x}{m} = \frac{15}{(3+4+8)} = 1.0$$

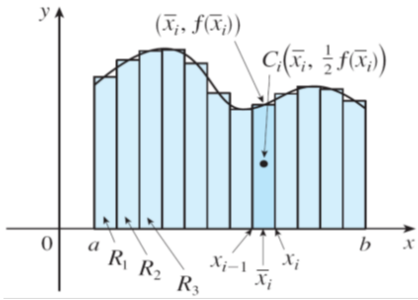
The center of mass $(\bar{x}, \bar{y}) = (1.93, 1)$

EXAMPLE 9.15 (CENTROID OF A LAMINA)

- What about the center of mass of a flat plate (lamina) that occupies the region $\mathcal{R}$ on a plane?
- The center of mass of the plate is called the centroid of $\mathcal{R}$
- Use the symmetry principle: if $\mathcal{R}$ is symmetric about a line l, then the centroid of $\mathcal{R}$ lies on l
- Example: what is the centroid of a rectangle?
- Next, define moments such that if the entire mass of a region is concentrated at the center of mass, its moments remain unchanged.



- Suppose that the region $\mathcal{R}$ lies between the lines *x* = *a* and *x* = *b*, above the *x*-axis, and beneath the graph of *f*, where *f* is a continuous function
- divide the interval [*a*, *b*] into *n* subintervals with endpoints x_0, $x_1, \ldots, x_n$ and equal width Δx
- choose the sample point x_i^* to be the midpoint $\bar{x}_i$ of the *i* th subinterval, that is, $\bar{x}_i = (x_{i-1} + x_i)/2$



- The centroid of the *i* th approximating rectangle R_i is its center $C_i\left(\bar{x}_i, \frac{1}{2}f(\bar{x}_i)\right)$. Its area is $f(\bar{x}_i)\Delta x$.
- The mass of the lamina is thus $\rho f(\bar{x}_i)\Delta x$
- The moment of R_i about the *y*-axis is the product of its mass and the distance from C_i to the *y*-axis, i.e. $\bar{x}_i$. Thus,

$$M_y(R_i) = [\rho f(\bar{x}_i)\Delta x]\bar{x}_i = \rho \bar{x}_i f(\bar{x}_i)\Delta x$$

- Adding these moments, we obtain the moment of the polygonal approximation to **R**
- By taking the limit as $n \to \infty$, we obtain the moment of **R** about the *y*-axis,

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x = \rho \int_a^b x f(x)\, dx$$

- Similarly, we compute the moment of R_i about the *x*-axis as the product of its mass and the distance from C_i to the *x*-axis:

$$M_x(R_i) = [\rho f(\bar{x}_i)\Delta x]\frac{1}{2} f(\bar{x}_i) = \rho \cdot \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x$$

- Adding the moments and take the limit to obtain the moment of **R** about the *x*-axis:

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x = \rho \int_a^b \frac{1}{2}[f(x)]^2\, dx$$

Recall the center of mass for a system of particles:

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

For a plate, the mass is

$$m = \rho A = \rho \int_a^b f(x) dx$$

Thus,

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx} = \frac{1}{A}\int_a^b x f(x)\, dx$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\, dx}{\int_a^b f(x)\, dx} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx$$

9.6 ARC LENGTH

Let consider the following arc,

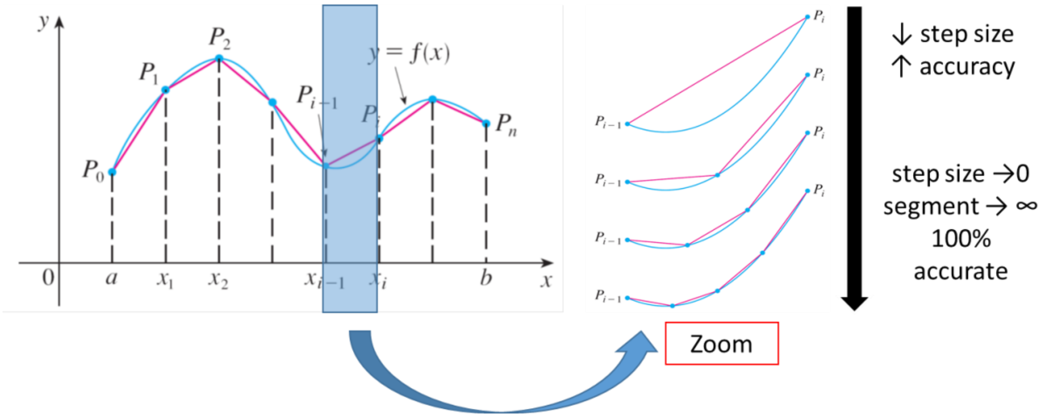


Figure 9.19

We want to determine the length of the continuous function y=f(x) on the interval [a,b]. We'll also need to assume that the derivative is continuous on [a, b].

Initially we'll need to estimate the length of the curve. We'll do this by dividing the interval up into n equal subintervals each of width Δx and we'll denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points.

Now denote the length of each of these line segments by $|P_{i-1,}\ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1} - P_i|$$

And we can get the exact length by taking n larger and larger, towards infinity. In other words, the exact length will be,

$$L = \lim_{n \to \propto} \sum_{i=1}^{n} |P_{i-1} - P_i|$$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define,

$$\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1}).$$

We can then compute directly the length of the line segments as follows:

$$|P_{i-1}\ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

$$|P_i - P_{i-1}| = \left(\sqrt{1 + (f'(x*))^2}\right) \Delta x \quad or, \quad = (\sqrt{1 + (\frac{1}{f'(x*)})^2}) \Delta y$$

The exact length of the curve is then,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1} \ P_i|$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2} \ \Delta x$$

However, using the definition of the definite integral, this is nothing more than,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

A slightly more convenient notation (in our opinion anyway) is the following.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

In a similar fashion we can also derive a formula for x=h(y) on [c, d]. This formula is,

$$L = \int_c^d \sqrt{1 + [h'(y)]^2} \, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

EXAMPLE 9.16

Find the length of the arc, $y = \frac{x^3}{3} + \frac{1}{4x}, 1 \ll x \ll 2$ as shown in Figure 9.13

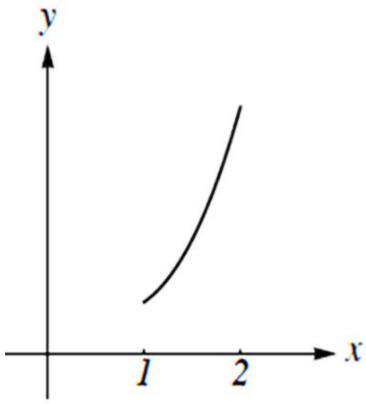


Figure: 9.20

Solution:

The derivative is,

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

We calculate a perfect square as follows:

$$\begin{aligned}\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \sqrt{1+\left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right)} \\ &= \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}} \\ &= \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} = x^2 + \frac{1}{4x^2}\end{aligned}$$

Then we obtain,

$$\begin{aligned}\text{(Arc length)} &= \int_1^2 \sqrt{1+[f'(x)]^2}\,dx = \int_1^2 \left(x^2 + \frac{1}{4x^2}\right) dx \\ &= \left.\frac{x^3}{3} - \frac{1}{4x}\right|_1^2 \\ &= \frac{59}{24}\end{aligned}$$

EXAMPLE 9.17

Find the arc length of $f(x) = \frac{x^2}{8} - lnx, 1 \leq x \leq e$ as shown in Figure 9.14

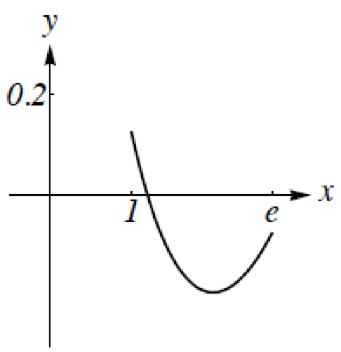


Figure 9.21

Solution:

The derivative is,

$$\frac{dy}{dx} = \frac{x}{4} - \frac{1}{x}.$$

The integrand of the arc length integral is given by,

$$\begin{aligned}\sqrt{1+(f'(x))^2} &= \sqrt{1+\left(\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}\right)} \\ &= \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}} \\ &= \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2} \\ &= \frac{x}{4}+\frac{1}{x} \qquad \text{since } x>0\end{aligned}$$

Then we find,

$$\begin{aligned}(\text{Arc length}) &= \int_1^e \sqrt{1+[f'(x)]^2}\,dx = \int_1^e \left(\frac{x}{4}+\frac{1}{x}\right)dx \\ &= \left.\frac{x^2}{8}+\ln x\right|_1^e \\ &= \frac{e^2+7}{8}.\end{aligned}$$

9.7 THE AREA OF A SURFACE OF REVOLUTION

Let C be a right circular cone with a base radius of r and slant height h, see Figure 9.21. If we slice open the cone at its vertex along a slant height and lay it on a plane, we obtain a sector of a circle of radius h and an intercepted arc of length 2πr. Then the central angle θ of the sector satisfies:

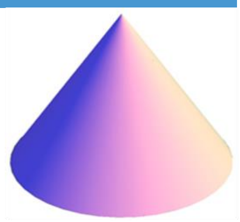


Figure 9.22: A right circular cone with slant height "h" and radius "r".

$$h\theta = 2\pi r$$

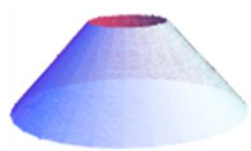


Figure 9.23: A frustum of cone with radii r2 > r1 and slant height "l".

Consequently, the area of the sector (or equivalently the surface area of C) is,

$$(Surface\ area\ of\ a\ cone) = \frac{\theta}{2}h^2 = \pi r h \qquad (9.4)$$

Next, consider a frustum of a cone with radii r2 > r1 and slant height l, see Figure 9.22. Using similar right triangles, as shown in Figure 9.23, where,

$$l = l_2 - l_1$$

We get,

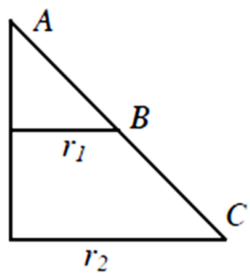


Figure 9.24: Similar triangles where AB =l1, AC = l2

$$\frac{l_2}{r_2} = \frac{l_1}{r_1}$$

Note, the surface area of a frustum is the difference of the surface areas of two cones. Applying formula (9.4), we obtain,

$$\begin{aligned}
\text{(Surface area of a frustum)} &= \pi r_2 l_2 - \pi r_1 l_1 = \pi(r_2 l_2 - r_1 l_1) \\
&= \pi(r_2 + r_1)(l_2 - l_1) \quad \text{Since } r_1 l_2 - r_2 l_1 = 0 \\
&= 2\pi r\ l. \quad \text{where } r = \frac{r_1 + r_2}{2}
\end{aligned}$$

EXAMPLE 9.18

Compute the surface area of a sphere of radius r. The sphere can be obtained by rotating the graph of $f(x) = \sqrt{r^2 - x^2}$ about the x-axis.

Solution:

The derivative f′ is $-x/\sqrt{r^2 - x^2}$, so the surface area is given by,

$$\begin{aligned}
A &= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\
&= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\, dx \\
&= 2\pi \int_{-r}^{r} r\, dx = 2\pi r \int_{-r}^{r} 1\, dx = 4\pi r^2
\end{aligned}$$

If the curve is rotated around the y axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn't change. Instead of the radius $f(\dot{x}_i^*)$, we use the new radius $\overline{x_i}$ = $(x_i - x_{i+1})/2$, and the surface area integral becomes,

$$\int_a^b 2\pi x\sqrt{1 + (f'(x))^2}\, dx.$$

EXAMPLE 9.19

Compute the area of the surface formed when f(x) = x^2 between 0 and 2 is rotated around the y-axis.

Solution:

We compute f’(x) = 2x, and then,

$$2\pi \int_0^2 x\sqrt{1+4x^2}\,dx = \frac{\pi}{6}(17^{3/2}-1)$$

by a simple substitution.